

Self force on an accelerated particle

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(Dated: December 11, 2013)

Abstract

We calculate the singular field of an accelerated point particle (scalar charge, electric charge or small gravitating mass) moving on an accelerated (non-geodesic) trajectory in a generic background spacetime. Using a mode-sum regularization scheme, we obtain explicit expressions for the self-force regularization parameters. In the electromagnetic and gravitational case, we use a Lorenz gauge. This work extends the work of Barack and Ori [1] who demonstrated that the regularization parameters for a point particle in geodesic motion in a Schwarzschild spacetime can be described solely by the leading and subleading terms in the mode-sum (commonly known as the A and B terms) and that all terms of higher order in ℓ vanish upon summation (later they showed the same behavior for geodesic motion in Kerr [2], [3]). We demonstrate that these properties are universal to point particles moving through any smooth spacetime along arbitrary (accelerated) trajectories. Our renormalization scheme is based on, but not identical to, the Quinn-Wald axioms. As we develop our approach, we review and extend work showing that different definitions of the singular field used in the literature are equivalent to our approach. Because our approach does not assume geodesic motion of the perturbing particle, we are able use our mode-sum formalism to explicitly recover a well-known result: The self-force on static scalar charges near a Schwarzschild black hole vanishes.

PACS numbers: 04.30.Db, 04.25.Nx, 04.70.Bw

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I. INTRODUCTION

The likelihood that the gravitational radiation from stellar-size black holes spiraling in to a supermassive galactic black will be observable has spurred work on the extreme-mass-ratio inspiral problem and on the analogous problem of a point particle with scalar or electric charge moving in a curved spacetime. The trajectory of a small body moving in a curved spacetime deviates from the geodesic motion of a point particle at linear order in the charge or mass. Derivations of the trajectory use matched asymptotic expansions and a point-particle limit of a family of finite bodies whose charge, mass and radius simultaneously shrink to zero; they show that one can describe this first-order trajectory by a renormalized self-force.¹ In an initial *MiSaTaQuWa* form developed for the metric perturbation of a massive particle by Mino, Sasaki and Tanaka [8] and by Quinn and Wald [9] – and for scalar fields by Quinn [10] – one uses the Hadamard expansion of the retarded Green’s function to identify a singular part of the field and a corresponding singular part f_α^{sing} of the expression f_α^{ret} for the particle’s self-force written in terms of the retarded field.

To subtract the singular from the retarded expression for the self-force, one first regulates each. This can in principle be done, as described in the *MiSaTaQuWa* papers, by evaluating them at a finite proper distance ρ from the trajectory and then taking a limit of their difference as $\rho \rightarrow 0$. Nearly all explicit calculations of the self-force on particles moving in Kerr or Schwarzschild geometries, however, have used a mode-sum form of the renormalization introduced by Barack and Ori [1, 11], with early development and first applications by them, Mino, Nakano, and Sasaki and Burko [12–14]. Its subsequent development and applications by a number of researchers are reviewed by Barack [3] and Poisson et al. [7]. In mode-sum regularization, one writes f_α^{sing} and f_α^{ret} as sums of angular harmonics on a sphere through the particle, replacing the short-distance cutoff ρ by a cutoff ℓ_{max} in the ℓ, m harmonics, and expressing the renormalized self-force as a limit $\lim_{\ell_{max} \rightarrow \infty} \left(\sum_{\ell=0}^{\ell_{max}} f_\alpha^{ret, \ell} - \sum_{\ell=0}^{\ell_{max}} f_\alpha^{sing, \ell} \right)$ or, equivalently, as the convergent sum $\sum_{\ell=0}^{\infty} (f_\alpha^{ret, \ell} - f_\alpha^{sing, \ell})$.

In this paper we generalize the results of Barack and Ori [1–3] for geodesic motion in a

¹ The most recent and rigorous of these are by Gralla, Harte, and Wald [4, 5] (with a formal proof for an electromagnetic charge), by Pound [6], and by Poisson, Pound and Vega [7], who also review the history and give a comprehensive bibliography.

Schwarzschild or Kerr background to accelerated trajectories in generic spacetimes, checking the method by computing the known (vanishing) self-force on a scalar particle at rest in a Schwarzschild background [15]. A striking feature of mode-sum regularization is that only the leading and subleading terms in ℓ^{-1} give nonzero contributions to the singular expression for the self-force: For a point particle with scalar charge, and, in a Lorenz gauge, for an electric charge and a point mass, $f_\alpha^{sing,\ell}$ has the form

$$f_\alpha^{sing,\ell\pm} = \pm A_\alpha L + B_\alpha, \quad (1)$$

where $L = \ell + 1/2$, A_α and B_α are independent of ℓ , and the sign \pm refers to a limit of the direction-dependent singular expression taken as one approaches the sphere through the particle from the outside or inside.²

The form of Eq. (1) describes the large ℓ behavior of f_α^{sing} and depends only on the short distance behavior of the retarded field. The values of the vectors A_α and B_α also depend on the choice of spherical coordinates in a neighborhood of the particle. For the electromagnetic and gravitational cases, the definition of f_α^{ret} involves, in addition to the retarded field (the vector potential A_α^{ret} or the perturbed gravitational field $h_{\alpha\beta}^{ret}$), the background metric $g_{\alpha\beta}$ and the particle's 4-velocity u^α , each evaluated at the particle's position $z(0)$. An expression for f_α^{ret} in the neighborhood of the particle then depends on how one extends $g_{\alpha\beta}[z(0)]$ and $u^\alpha[z(0)]$ to the neighborhood. Different smooth extensions leave the form of Eq. (1) and the value of the vector A_α unchanged, but they change the value of the subleading term, B_α .

To obtain Eq. (1) and the values of A_α and B_α , one arbitrarily extends f_α^{sing} from a normal neighborhood of the particle to a thick sphere spanned by spherical coordinates, but the values of A_α and B_α do not depend on that extension. Although the coefficient of any finite angular harmonic does depend on the extension, two different extensions that are smooth outside the normal neighborhood differ only by a smooth function; coefficients of the angular harmonics of a smooth function on the sphere fall off faster than any power of ℓ .

In the next section, we also show the equivalence of several renormalization methods: renormalization using mode-sum regularization, using regularization based on a short-

² Although the mode-sum expansion of the retarded field has terms of higher powers in ℓ^{-1} , we show that the sum of these terms vanishes. In computing the self-force, however, one improves convergence by explicitly including higher-order terms in the singular field.

distance cutoff, and related versions of renormalization that involve an angle average over a sphere of radius ρ about the particle. We use the Detweiler-Whiting form of the singular field [16] as a smooth, locally defined solution to simplify the analysis. This section is primarily a review, but the discussions of equivalence do not appear in one place in the literature and are often restricted to geodesic motion.

Generalizing the mode-sum formalism to accelerated trajectories allows one to use the method to find the self-force on charged, massive particles in spacetimes with background scalar or electromagnetic fields. The extension also allows one to consider flat-space limits of bound orbits in which the size of the orbit remains finite. It simplifies the derivation of the self-force on a static particle, as we show in Sec. V. For the gravitational case, a consistent treatment of an accelerated particle of mass m must include the matter responsible for the acceleration. Because our computation of the gravitational self-force includes only the contribution from the particle, it does not by itself describe the correction to the orbit of an accelerated mass at order m . The gravitational and electromagnetic contributions to the singular part of the self-force, however, have a natural form as a sum of the contributions we obtain here and contributions arising at subleading order from terms that couple the two fields [17, 18]. We mention in Sec. IV B our work in progress on the self-force on a charged, massive particle in an electrovac spacetime. The study may be useful in deciding whether including the self-force prevents one from overcharging a near-extremal black hole [19–22].

Our renormalization scheme and connection with previous results

Our renormalization scheme is a slight modification of the method used by Quinn [10]. However, in the process of carrying out the calculation, we can show the equivalence of our technique with that of Detweiler and Whiting [16].

Quinn uses the DeWitt-Brehme [23] formalism to expand the gradient of the retarded field in close proximity to the scalar charge. This gives a highly divergent Coulomb field (*i.e.* a term of $O(\epsilon^{-2})$, where ϵ is a measure of the distance from the particle) and another divergent term proportional to the acceleration which scales as $O(\epsilon^{-1})$. Some non-divergent terms proportional to the square of the acceleration and the curvature tensors appear at $O(\epsilon^0)$. Also appearing at $O(\epsilon^0)$ are the terms that actually produce the self-force: the \ddot{x} terms which give the standard Abraham-Lorentz-Dirac force, as well as an integral over

the past history (*i.e.* the tail term). Quinn notes that the first terms (the Coulomb, the acceleration and acceleration-squared terms) are the same as those that are present in flat spacetime for the half-advanced plus half-retarded field.

Quinn's second axiom asserts that a charge moving in flat spacetime with a half-advanced plus half-retarded field will experience no self-force. His first axiom allows him to subtract the flat spacetime terms from the curved spacetime field without modifying the resulting force. The subtraction removes all singular terms because the singular terms are the same in flat spacetime as they are in curved space. The subtraction leaves only the curvature terms and those terms actually responsible for the force. His second axiom also requires an angle average over a sphere near the charge. This angle average removes the curvature terms and leaves only those terms which contribute to the self-force.

This is an elegant procedure; however the final step of angle-averaging over a small ball centered on the charge is difficult to carry out when the fields are computed using a mode-sum with coordinates centered on the black hole.

To get past this technical difficulty, we modify Quinn's prescription. When we do the subtraction, we also subtract the curvature terms. This eliminates all the divergent terms and it eliminates the need for angle-averaging. The only terms that remain are those that contribute to the self-force.

Detweiler and Whiting arrive at a similar prescription, but through a somewhat different argument. The connection between the methods can be explained as follows: The quantity they subtract from the full retarded field gradient is the gradient of their $\Psi^S(x)$ (*i.e.* Eq. (17) of [16]). By expanding this vector and keeping terms of $O(\epsilon^{-2})$, $O(\epsilon^{-1})$ and $O(\epsilon^0)$, one arrives at precisely the same quantity we subtract from the retarded field gradient: the Coulomb field, the acceleration terms, the acceleration-squared terms, and the Riemann curvature terms.

Though many of the steps in this paper are closely related to other work, the over-riding intent of this paper is to give a self-contained discussion of the self-force problem: We start with the field equations and end with the regularization parameters (the A and B terms) needed for a mode-sum calculation of the finite part of the self-force for an accelerated point particle moving in an arbitrary spacetime.

Why do we take the time to derive the field gradient expressions anew? Why don't we shorten the paper and simply take the expressions from Quinn [10] (Or Poisson, Pound

and Vega [7] see their Eqs. (17.39) and (18.20))? There are two (related) reasons why we do not do this. First, we wish to exploit the techniques of Barack and Ori to derive the regularization parameters; therefore we choose to express the field gradient with a notation and quasi-Cartesian coordinate system that is essentially identical to theirs, allowing us to use their methods directly. At the very least, using previously obtained results for the field gradients would require a messy notation or coordinate conversion in order to write them in a usable form for our calculations, or, alternatively, recasting the techniques of Barack and Ori into a Quinn-like notation. Simply rederiving the field gradients in our notation is the most straight-forward and illuminating path.

A second reason for not simply lifting the expressions for the field gradients from the previous literature is that such expressions really do not contain sufficient information for our calculation. In Quinn, for example, the expression of the field gradient is given at a field point that lies along a spatial geodesic orthogonal to the world line; Quinn's expression relates the field gradient on this orthogonal spatial slice to the state of motion (acceleration and jerk) of the particle on this same spatial slice. In order to compute the mode-sums, we need the field gradient at points on a slice of constant coordinate time. Our gradient is expressed in term of the state of motion of the particle at that same instant of coordinate time. Thus the results given in [10] and [7] don't give us this necessary starting point for our calculation.³

We begin in Sect. II with the singular field of a scalar charge, using the results of Quinn [10] and Detweiler and Whiting [16], to express f_{α}^{sing} explicitly in Riemann normal coordinates (RNCs). In Sect. III, we show that $f_{\alpha}^{sing,\ell}$ has the form given in Eq. (1), and we find explicit expressions for the regularization parameters A_{α} and B_{α} . Next, in Sect. IV, we generalize this analysis to electric charges and masses in a Lorenz gauge. In each case, the mode-sum analysis is closely patterned on the work of Barack and Ori [24]. Finally, in Sect. V, we check the validity of the method by recovering the result of Wiseman [15] for a static charge in a Schwarzschild spacetime.

³ Of course one could, for example, use Quinn's expression to obtain the field gradient on a slice of constant time. Each point on our slice of constant time corresponds to a point on a particular slice that is orthogonal to the world line. However, this slice would intersect the world line at a different coordinate time. One then has to relate the motion of the particle as it passes through this orthogonal slice to the motion at the desired moment in coordinate time.

In this paper we will always use the conventions of Misner, Thorne and Wheeler, [25] unless otherwise noted.

II. SELF FORCE AND THE SINGULAR FIELD.

We consider a point particle (a scalar charge q , electric charge e , or mass m) traveling on an accelerated trajectory $z(\tau)$ in a smooth spacetime $(M, g_{\alpha\beta})$, where τ is proper time. We will use RNCs about a point $\tau = 0$ of the trajectory and, for mode-sum regularization, spherical coordinates (t, r, θ, ϕ) associated with an arbitrary smooth Cartesian chart. For brevity of notation, we assume that $t = 0$ at $\tau = 0$.

We consider a field point x that lies on the spacelike $t = 0$ slice and is in a convex normal neighborhood C of $z(0)$. We denote by ϵ the geodesic distance from the particle's position at an arbitrary time τ to x ; that is, ϵ is the length of the unique geodesic from an arbitrary point on the trajectory $z(\tau)$. After performing the various derivative operations to get to an expression for the singular field and singular force, we will choose our arbitrary point to be $z(0)$. In particular, when we reach the mode-sum section, we will consider ϵ to be the length of the unique geodesic from $z(0)$ to x (see Figure 1).

In this section we focus on the self-force on a scalar charge, but we will subsequently use the same mathematical framework to calculate the self-force for both electric charges and point masses. We begin with the MiSaTaQuWa renormalization of a scalar field, using the axiomatic renormalization description given by Quinn [10]. We use the Hadamard expansion of the advanced and retarded fields to show that this description is equivalent to subtracting a singular part f_α^{sing} of the retarded expression f_α^{ret} for the self-force, regularized by a short-distance cutoff. From the explicit form of f_α^{sing} , we show that, for geodesic motion, the renormalization is equivalent to an angle average of f_α^{ret} . We end the section with a check of the equivalence of the singular field seen in a Hadamard expansion and Detweiler-Whiting form.

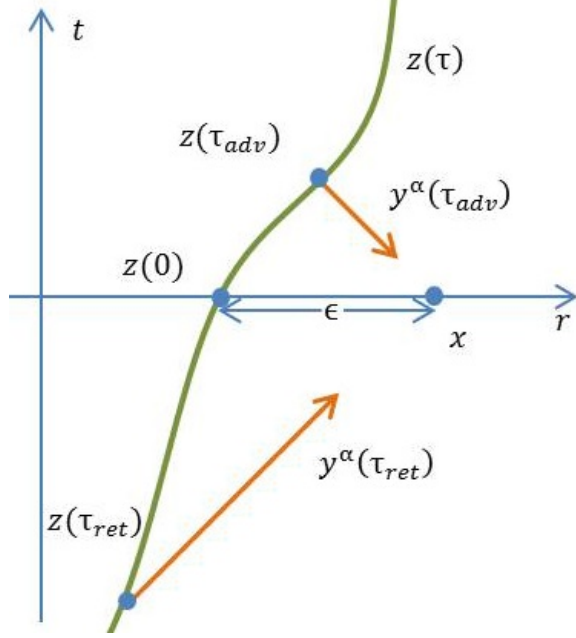


FIG. 1: The particle trajectory $z(\tau)$. Two null vectors $y^\alpha(\tau_{ret})$ and $y_\alpha(\tau_{adv})$ are tangent to future- and past-directed null geodesics from points along the trajectory to a field point x . A geodesic from $z(0)$ to x has length ϵ .

A. Self-Force on a Scalar Charge

The self-force correction to the equation of motion of a particle with charge q is $O(q^2)$. For matter with scalar charge density ρ , a zero-rest-mass scalar field Φ satisfies

$$\nabla_\alpha \nabla^\alpha \Phi = -4\pi\rho, \quad (2)$$

where, for a point-particle, the density ρ is given by the distribution

$$\rho(x) = q \int d\tau \delta^4(x, z(\tau)), \quad (3)$$

and the formal expression for the self-force,

$$f^\alpha = q \nabla^\alpha \Phi, \quad (4)$$

diverges at the position of the particle.

Quinn's description of the renormalized self-force is stated as two axioms, based on the Quinn-Wald axioms [9] for higher spins. To present the axioms, we introduce a set of RNCs whose origin is the position $z(0)$ of the particle at proper time $\tau = 0$. Coordinates and

components in these coordinates will be denoted by hatted indices. The coordinates of the particle's position are then $z^{\hat{\alpha}}(\tau)$, with $z^{\hat{\alpha}}(0) = 0$, and a field point x has coordinates $x^{\hat{\alpha}}$.

Quinn allows Φ to be any scalar field satisfying Eq. (2), with ρ given by Eq. (3). The field Φ can then include an arbitrary homogeneous background scalar field as well as the retarded field of the particle (if, in fact, a retarded field is well defined on the spacetime); the renormalized force, which we denote by $f_{Q\alpha}^{ren}$, then includes the force due to the homogeneous background field as well as the self-force. We initially state the axioms with this generality and then restrict consideration to the retarded field. With this restriction, the force $f_{Q\alpha}^{ren}$ is the self-force f_{α}^{ren} .

Quinn's first axiom, the comparison axiom can be stated as follows:

Consider two point particles in two possibly different spacetimes, each particle having scalar charge q . Suppose that, at points $z(0)$ and $\tilde{z}(0)$ on their respective trajectories, the magnitude of the particles' 4-accelerations coincide. We may then choose RNC systems about $z(0)$ and about $\tilde{z}(0)$ for which the components of the 4-velocities and 4-accelerations coincide:

$$u^{\hat{\alpha}} = \tilde{u}^{\hat{\alpha}}, \quad a^{\hat{\alpha}} = \tilde{a}^{\hat{\alpha}}. \quad (5)$$

Let Φ and $\tilde{\Phi}$ be the retarded scalar fields of the particles. With the RNC systems used to identify neighborhoods of $z(0)$ and $\tilde{z}(0)$, the difference between the renormalized scalar forces, $f_{Q\alpha}^{ren}$ and $\tilde{f}_{Q\alpha}^{ren}$ is given by the limit as $r \rightarrow 0$ of the gradients of the fields averaged over a sphere of geodesic distance r about $z(0)$.⁴

$$f_Q^{ren,\hat{\alpha}} - \tilde{f}_Q^{ren,\hat{\alpha}} = q \lim_{r \rightarrow 0} \langle \nabla^{\hat{\alpha}} \Phi - \nabla^{\hat{\alpha}} \tilde{\Phi} \rangle_r. \quad (6)$$

Quinn's second axiom simply states that the renormalized scalar force vanishes for the half-advanced + half-retarded field of a uniformly accelerated charge in flat space:

If, for a uniformly accelerated scalar charge in flat space, $\tilde{\Phi} = \frac{1}{2}(\tilde{\Phi}^{ret} + \tilde{\Phi}^{adv})$,
then $\tilde{f}_Q^{ren,\alpha} = 0$.

⁴ With S_r the set of points that lie a geodesic distance r from $z(0)$ along a geodesic perpendicular to the trajectory, the average of a function f is $\langle f \rangle_r := |S_r|^{-1} \int_{S_r} f dS$, where $|S_r|$ is the area of S_r .

To define the self-force, we assume that the spacetime of the field Φ is globally hyperbolic so that retarded and advanced fields are well defined, and we set $\Phi = \Phi^{ret}$. With this restriction, the axioms imply that the self-force is given by

$$f^{ren,\hat{\alpha}} = q \lim_{r \rightarrow 0} \langle \nabla^{\hat{\alpha}} \Phi^{ret} - \nabla^{\hat{\alpha}} \tilde{\Phi} \rangle_r. \quad (7)$$

As in this equation, we will henceforth use the RNC identification of normal neighborhoods of the flat and curved spacetimes to regard $\tilde{\Phi}$ as a field on C .

B. Short-Distance Expansion of the Retarded and Advanced Fields

While Quinn's description makes no explicit mention of a singular part of the scalar force, we will see that the result of the subtraction and angle average is equivalent to identifying and subtracting such a singular part: a vector field f_{α}^{sing} defined in a neighborhood of the particle. To do so, we partly follow Quinn, who in turn uses the DeWitt-Brehme formalism, to find the singular behavior of the advanced and retarded fields near the particle. Quinn obtains an expression for the field at points x lying on a geodesic on a spacelike hypersurface orthogonal to the trajectory at a point $z(0)$ in terms of the particle's velocity, acceleration and jerk at $z(0)$. For the mode-sum regularization of Sect. III, however, we need an expression for the field on a hypersurface that is not orthogonal to the trajectory. For this reason, we obtain in this section the field at an arbitrary nearby point x and use it to identify f_{α}^{sing} .

To rewrite f_{α}^{ren} of Eq. (7) as a difference of the form

$$f_{\alpha}^{ren} = \lim_{x \rightarrow z(0)} [f_{\alpha}^{ret}(x) - f_{\alpha}^{sing}(x)], \quad (8)$$

we restrict x to lie in the normal neighborhood C of $z(0)$. Because we have chosen C to be convex, any points $x, x' \in C$ are joined by a geodesic, and the advanced and retarded Green's functions have the Hadamard forms,

$$G^{adv/ret}(x, x') = \Theta_{\pm}(x, x') [U(x, x')\delta(\sigma(x, x')) - V(x, x')\theta(-\sigma(x, x'))]. \quad (9)$$

Here $V(x, x')$ and $U(x, x')$ are smooth bi-scalar functions of x and x' , and $\sigma(x, x')$ is half the squared length of the geodesic connecting x and x' . The function $\Theta_{\pm}(x, x')$ is unity when x' is in the causal future (past) of the event x for the advanced (retarded) Green's function, and vanishes otherwise.

The retarded solution to Eqs. (2) and (3) is given by

$$\begin{aligned}\Phi^{ret} &= q \int d^4x' \sqrt{-g} \int d\tau G^{ret}(x, x') \delta^4(x', z(\tau)), \\ &= q \int d\tau G^{ret}(x, z(\tau)).\end{aligned}\quad (10)$$

Following Quinn, we break the domain of integration into two regions: the part of the trajectory in the normal neighborhood C (where the Hadamard form of the Green's function is valid) and the rest of the trajectory. We choose the event x to be close enough to the trajectory that the events $z(\tau_{adv})$ and $z(\tau_{ret})$ both lie in C , and we denote by T_{\pm} the proper times at which the trajectory intersects the boundary ∂C : The past and future intersection points are respectively $z(T_-)$ and $z(T_+)$. The retarded field then takes the form

$$\begin{aligned}\Phi^{ret} &= q \int_{T_-}^{T_+} \Theta_-(x, z(\tau)) [U(x, z(\tau)) \delta(\sigma(x, z(\tau))) - V(x, z(\tau)) \theta(-\sigma(x, z(\tau)))] d\tau + q \int_{-\infty}^{T_-} G^{ret} d\tau, \\ &= q \int_{T_-}^0 [U \delta(\sigma) - V \theta(-\sigma)] d\tau + q \int_{-\infty}^{T_-} G^{ret} d\tau,\end{aligned}$$

where we have suppressed the arguments of the biscalar functions. Noting that in the interval $[T_-, 0]$, $\sigma(x, z(\tau)) = 0$ only at $\tau = \tau_{ret}$, and using $d\tau = \dot{\sigma}^{-1} d\sigma$, with $(\dot{}) = d/d\tau$, we have,

$$\Phi^{ret}(x) = q \left(\frac{U(x, z(\tau))}{\dot{\sigma}} \right)_{ret} - q \int_{T_-}^{\tau_{ret}(x)} V(x, z(\tau)) d\tau + q \int_{-\infty}^{T_-} G^{ret}(x, z(\tau)) d\tau. \quad (11)$$

The gradient of Φ with respect to x is given by

$$\nabla_{\alpha} \Phi^{ret} = q \nabla_{\alpha} \left[\left(\frac{U}{\dot{\sigma}} \right)_{ret} \right] + q V \nabla_{\alpha} \tau_{ret} - q \int_{T_-}^{\tau_{ret}} \nabla_{\alpha} V d\tau + q \int_{-\infty}^{T_-} \nabla_{\alpha} G^{ret} d\tau. \quad (12)$$

Because $\nabla_{\alpha} V(x, z(\tau))$ and $\nabla_{\alpha} G^{ret}(x, z(\tau))$ are vectors in the tangent space at x for all values of τ , the integrals are well defined.

Noticing that, for $T_- \leq \tau < \tau_{ret}$, $G^{ret}(x, z(\tau)) = -V(x, z(\tau))$, we write

$$\nabla_{\alpha} \Phi^{ret} = q \nabla_{\alpha} \left[\left(\frac{U}{\dot{\sigma}} \right)_{ret} \right] + q V \nabla_{\alpha} \tau_{ret} + q \lim_{h \rightarrow 0} \int_{-\infty}^{\tau_- - h} \nabla_{\alpha} G^{ret} d\tau. \quad (13)$$

We can also write the retarded and advanced solutions to the field equation as

$$\Phi^{ret/adv} = q \left[\frac{U(x, z)}{\dot{\sigma}} \right]_{ret/adv} \pm q \lim_{h \rightarrow 0} \int_{\mp\infty}^{\tau_{ret/adv} \mp h} G^{ret/adv}(x, z) d\tau, \quad (14)$$

and

$$\nabla_{\alpha} \Phi^{ret/adv} = q \nabla_{\alpha} \left[\frac{U(x, z)}{\dot{\sigma}} \right]_{ret/adv} \pm q V(x, z) \nabla_{\alpha} \tau_{ret/adv} \pm q \lim_{h \rightarrow 0} \int_{\mp\infty}^{\tau_{ret/adv} \mp h} \nabla_{\alpha} G^{ret/adv}(x, z) d\tau. \quad (15)$$

Now that we have an expression for the retarded and advanced forces, we need to find expansions of the three bi-scalars, U , V , and σ .

1. *Expanding the Biscalars, $U(x, z)$, $V(x, z)$, and $\sigma(x, z)$*

The quantities $U(x, z)$ and $V(x, z)$ have the local expansions [23]

$$U(x, z) = 1 + \frac{1}{12} R_{\alpha'\beta'} \nabla^{\alpha'} \sigma(x, z) \nabla^{\beta'} \sigma(x, z) + O(\epsilon^3), \quad (16)$$

$$V(x, z) = -\frac{1}{12} R(z) + O(\epsilon), \quad (17)$$

where $\nabla^{\alpha'}$ is defined to be the contravariant derivative at the position of the particle (z), $R_{\alpha\beta}$ is the Ricci Tensor, and $R(z)$ is the Ricci Scalar.

We review here the computation that expresses $\dot{\sigma}_{ret/adv}$ in terms of the coordinates $x^{\hat{\alpha}}$, and the particle's 4-velocity u^α , acceleration a^α , and jerk $\dot{a}^\alpha := u^\beta \nabla_\beta a^\alpha$ at $\tau = 0$. We write $\dot{\sigma}_{ret/adv} = -(u^\alpha y_\alpha)_{ret/adv}$, where $-y_{\alpha, ret}$ and $-y_{\alpha, adv}$ are the gradients with respect to z of $\sigma(x, z)$ at $z_{ret} = z(\tau_{ret})$ and $z_{adv} = z(\tau_{adv})$,

$$y_{\alpha, ret/adv} := -(\nabla_\alpha \sigma)_{ret/adv}. \quad (18)$$

The contravariant vectors $y_{ret/adv}^\alpha$ are tangent to affinely parameterized null geodesics from $z(\tau_{ret/adv})$ to x . Solving the geodesic equation iteratively, we find

$$y_{ret}^{\hat{\alpha}} = (x^{\hat{\alpha}} - z_{ret}^{\hat{\alpha}}) - \frac{1}{3} R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}\hat{\gamma}} z_{ret}^{\hat{\gamma}} (x^{\hat{\mu}} - z_{ret}^{\hat{\mu}}) (x^{\hat{\nu}} - z_{ret}^{\hat{\nu}}) + O(\epsilon^4). \quad (19)$$

For the advanced term, $y_{adv}^{\hat{\alpha}}$, replace each subscript “ret” by “adv”. We next expand $z^{\hat{\alpha}}(\tau)$ about $\tau = 0$:

$$z^{\hat{\alpha}}(\tau_{ret/adv}) = z^{\hat{\alpha}}(0) + \partial_\tau z^{\hat{\alpha}}|_{\tau=0} \tau_{ret/adv} + \frac{1}{2} \partial_\tau^2 z^{\hat{\alpha}}|_{\tau=0} \tau_{ret/adv}^2 + O(\tau^3). \quad (20)$$

Using the form of the Christoffel symbols in RNC, $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} = -\frac{2}{3} R^{\hat{\alpha}}{}_{(\hat{\beta}\hat{\gamma})\hat{\delta}} x^{\hat{\delta}}$, and the index symmetries of the Riemann tensor, we have

$$a^{\hat{\alpha}} = u^{\hat{\beta}} \nabla_{\hat{\beta}} u^{\hat{\alpha}}|_{\tau=0} = \partial_\tau^2 z^{\hat{\alpha}}|_{\tau=0}, \quad \dot{a}^{\hat{\alpha}} = u^{\hat{\beta}} \nabla_{\hat{\beta}} a^{\hat{\alpha}}|_{\tau=0} = \partial_\tau^3 z^{\hat{\alpha}}|_{\tau=0}, \quad (21)$$

whence

$$z^{\hat{\alpha}}(t_{ret/adv}) = u^{\hat{\alpha}} \tau_{ret} + \frac{1}{2} a^{\hat{\alpha}} \tau_{ret}^2 + \frac{1}{6} \dot{a}^{\hat{\alpha}} \tau_{ret}^3 + O(\tau_{ret}^4), \quad (22)$$

with each coefficient evaluated at $\tau = 0$. Now we use the relation $(g_{\alpha\beta} y^\alpha y^\beta)_{ret/adv} = 0$ to find $\tau_{ret/adv}$ in terms of $u^{\hat{\alpha}}$ and $x^{\hat{\alpha}}$. Writing $\tau_{ret/adv} = \tau_1 + \tau_2 + O(\tau^3)$, with $\tau_n = O(\epsilon^n)$, we find

$$\tau_1 = - \left(u_{\hat{\alpha}} x^{\hat{\alpha}} \pm \sqrt{(\eta_{\hat{\alpha}\hat{\beta}} + u_{\hat{\alpha}} u_{\hat{\beta}}) x^{\hat{\alpha}} x^{\hat{\beta}}} \right), \quad (23)$$

where the \pm corresponds to retarded (+) and advanced (-) solutions and $u_{\hat{\alpha}}$ is evaluated at $\tau = 0$. We denote by

$$q_{\alpha\beta} := g_{\alpha\beta} + u_{\alpha}u_{\beta} \quad (24)$$

the projection operator orthogonal to u_{α} and, with notation motivated by Eq. (31) below, write $\hat{S}_0 = q_{\hat{\alpha}\hat{\beta}}x^{\hat{\alpha}}x^{\hat{\beta}}$, where $q_{\hat{\alpha}\hat{\beta}}$ is evaluated at $z(0)$. Then

$$\tau_1 = - \left(u_{\hat{\mu}}x^{\hat{\mu}} \pm \sqrt{\hat{S}_0} \right). \quad (25)$$

Similarly,

$$\tau_2 = \pm \frac{a_{\hat{\alpha}}x^{\hat{\alpha}}}{2\sqrt{\hat{S}_0}} \tau_1^2. \quad (26)$$

Finally, by substituting Eqs. (23), and (26) into Eq. (22) we obtain an expression for $z_{ret/adv}^{\hat{\alpha}}$ (and thus $y^{\hat{\alpha}}$) entirely in terms of $x^{\hat{\alpha}}$ and of $u^{\hat{\alpha}}$ and their derivatives at $t = 0$.

We now expand $\dot{\sigma}$ about $\epsilon = 0$. To do this, we focus on $\dot{\sigma}^2$ and pattern our calculation on that of [1]. Thus, we write,

$$\dot{\sigma}_{ret/adv}^2 = (u_{\hat{\alpha}}y^{\hat{\alpha}})_{ret/adv}^2 = \left(q_{\hat{\alpha}\hat{\beta}}y^{\hat{\alpha}}y^{\hat{\beta}} \right)_{ret/adv}. \quad (27)$$

Here u^{α} is the four velocity of the particle at the retarded or advanced times (we treat this in a similar manner to the way we treated $z_{ret/adv}^{\hat{\alpha}}$, using a similar expansion as in Eq. (22)). Since $y_{ret/adv}^{\alpha}$ is a null vector, we were able to add the term $g_{\hat{\alpha}\hat{\beta}}y^{\hat{\alpha}}y^{\hat{\beta}} = 0$. The reason for this change will soon be clear.

To keep track of the relevant terms in the calculation, we borrow a term from [1], and then generalize it. We define \hat{S} by ⁵

$$\hat{S} := \left[q_{\hat{\alpha}\hat{\beta}}(x^{\hat{\alpha}} - z^{\hat{\alpha}})(x^{\hat{\beta}} - z^{\hat{\beta}}) \right]_{ret/adv}. \quad (28)$$

With this definition, we can now write

$$\dot{\sigma}_{ret/adv}^2 = S_{ret/adv} + \frac{1}{3}R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}}x^{\hat{\alpha}}x^{\hat{\beta}}u^{\hat{\gamma}}u^{\hat{\lambda}}(x^{\hat{i}}x_{\hat{i}}) + O(\epsilon^5). \quad (29)$$

Here and in the rest of this section, $q_{\hat{\alpha}\hat{\beta}}, u^{\hat{\alpha}}, a^{\hat{\alpha}}$, and $\dot{a}^{\hat{\alpha}}$ will all be assumed to be evaluated at $\tau = 0$. When we expand S about $\epsilon = 0$, we find

$$\hat{S} = \hat{S}_0 + \hat{S}_1 + \hat{S}_2 + \dots \quad (30)$$

⁵ It is useful to note that in [3] the use of the hat denoted a quantity evaluated at $\delta r = 0$, whereas we use hats to specify that the expression is one found using RNCs. When we need to make a similar evaluation we will denote these quantities with a tilde.

where $\hat{S}_n = O(\epsilon^{n+2})$. Explicitly, we have

$$\hat{S}_0 = (\eta_{\hat{\alpha}\hat{\beta}} + u_{\hat{\alpha}}u_{\hat{\beta}})x^{\hat{\alpha}}x^{\hat{\beta}}, \quad (31)$$

$$\hat{S}_1 = \eta_{\hat{\alpha}\hat{\beta}}a_{\hat{\gamma}}x^{\hat{\alpha}}x^{\hat{\beta}}x^{\hat{\gamma}}, \quad (32)$$

and

$$\hat{S}_2 = S_2^{(1)} \pm S_2^{(\pm)} = \left[\Sigma_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}}^{(1)} \pm \frac{x^{\hat{\delta}}}{\sqrt{\hat{S}_0}} \Sigma_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}\hat{\delta}}^{(\pm)} \right] x^{\hat{\alpha}}x^{\hat{\beta}}x^{\hat{\gamma}}x^{\hat{\lambda}}, \quad (33)$$

where the quantities $\Sigma_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}}^{(1)}$ and $\Sigma_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}\hat{\delta}}^{(\pm)}$ in Eq. (33) are

$$\Sigma_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}}^{(1)} := \frac{a^2}{12}q_{\hat{\alpha}\hat{\beta}} \left((\eta_{\hat{\gamma}\hat{\lambda}} + 7u_{\hat{\gamma}}u_{\hat{\lambda}}) - u_{\hat{\alpha}}u_{\hat{\beta}}u_{\hat{\gamma}}u_{\hat{\lambda}} \right) - \frac{u_{\hat{\lambda}}\dot{a}_{\hat{\gamma}}}{3}(3\eta_{\hat{\alpha}\hat{\beta}} + 2u_{\hat{\alpha}}u_{\hat{\beta}}) \quad (34)$$

and

$$\Sigma_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}\hat{\delta}}^{(\pm)} := \frac{2}{3}(\eta_{\hat{\alpha}\hat{\beta}} + u_{\hat{\alpha}}u_{\hat{\beta}})(\eta_{\hat{\gamma}\hat{\lambda}} + u_{\hat{\gamma}}u_{\hat{\lambda}})(a^2u_{\hat{\delta}} - \dot{a}_{\hat{\delta}}). \quad (35)$$

It is also useful to define

$$r_{\hat{\alpha}} := \frac{1}{2}\nabla_{\hat{\alpha}}\hat{S}_0 = \nabla_{\hat{\alpha}}(\eta_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})x^{\hat{\mu}}x^{\hat{\nu}} = (\eta_{\hat{\alpha}\hat{\mu}} + u_{\hat{\mu}}u_{\hat{\alpha}})x^{\hat{\mu}}. \quad (36)$$

We now have the information to write the expansion of the first term in Eq. (14) (sometimes called the ‘direct’ term). We use Eqs. (16), (29), (30), (31), (32), and (33) to expand $\Phi^{ret/adv}$ to the first three orders in ϵ :

$$\begin{aligned} \Phi^{ret/adv} &= \frac{q}{\sqrt{\hat{S}_0}} \left[1 - \frac{\hat{S}_1}{2\hat{S}_0} + \frac{3}{8} \left(\frac{\hat{S}_1}{\hat{S}_0} \right)^2 - \frac{\hat{S}_2}{2\hat{S}_0} \right] - \frac{q}{6\hat{S}_0^{3/2}} R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}} u^{\hat{\lambda}} u^{\hat{\gamma}} x^{\hat{\alpha}} x^{\hat{\beta}} x^2 \\ &+ \frac{qR_{\hat{\alpha}\hat{\beta}}}{12} \left[\frac{r^{\hat{\alpha}}r^{\hat{\beta}} + S_0u^{\hat{\alpha}}u^{\hat{\beta}}}{\sqrt{\hat{S}_0}} \pm 2(x^{\hat{\alpha}}u^{\hat{\beta}} + u^{\hat{\alpha}}u^{\hat{\beta}}u_{\hat{\gamma}}x^{\hat{\gamma}}) \right] \pm q \lim_{h \rightarrow 0} \int_{\mp\infty}^{\tau_{ret/adv} \mp h} G^{ret/adv}(x, z) d\tau + O(\epsilon^2), \end{aligned} \quad (37)$$

where $x^2 \equiv x^{\hat{\epsilon}}x_{\hat{\epsilon}}$.

Because the flat spacetime Green’s function has support only on the light cone and not in its interior, the last term in (35) vanishes in flat spacetime, as do the terms involving the Ricci tensor. On the other hand, the curved spacetime and flat spacetime values of \hat{S}_0 , \hat{S}_1 and \hat{S}_2 coincide. The flat-space comparison field $\tilde{\Phi}$ of Eq. (7) is thus obtained from the first term of Eq. (37) by taking half the sum of its retarded and advanced forms. Noting that

\hat{S}_0 and \hat{S}_1 are the same for the retarded and advanced field and that, by Eq. (33), \hat{S}_2 is replaced by $\hat{S}_2^{(1)}$, we have

$$\tilde{\Phi} = \frac{q}{\sqrt{\hat{S}_0}} \left[1 - \frac{\hat{S}_1}{2\hat{S}_0} + \frac{3}{8} \left(\frac{\hat{S}_1}{\hat{S}_0} \right)^2 - \frac{\hat{S}_2^{(1)}}{2\hat{S}_0} \right]. \quad (38)$$

Note that the flat-space contribution $\tilde{\Phi}$ includes the leading singular behavior (the Coulomb part) of Φ^{ret} .

It is instructive to see Eq. (37) written in terms of the acceleration and jerk. Using Eqs. (31)-(35), we obtain

$$\begin{aligned} \Phi^{ret/adv} = & \frac{q}{\sqrt{q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}}} \left[1 - \frac{a_{\hat{\gamma}}x^{\hat{\gamma}}x^2}{2q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}} \left(1 - \frac{3}{4} \frac{a_{\hat{\gamma}}x^{\hat{\gamma}}x^2}{q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}} \right) \right] \mp \frac{q}{3} x^{\hat{\delta}} (a^2 u_{\hat{\delta}} - \dot{a}_{\hat{\delta}}) \\ & - \frac{q \left[a^2 \left(q_{\hat{\alpha}\hat{\beta}} (\eta_{\hat{\gamma}\hat{\lambda}} + 7u_{\hat{\gamma}}u_{\hat{\lambda}}) - u_{\hat{\alpha}}u_{\hat{\beta}}u_{\hat{\gamma}}u_{\hat{\lambda}} \right) - 4u_{\hat{\gamma}}a_{\hat{\lambda}}(3\eta_{\hat{\alpha}\hat{\beta}} + 2u_{\hat{\alpha}}u_{\hat{\beta}}) \right]}{24 (q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}})^{3/2}} x^{\hat{\alpha}}x^{\hat{\beta}}x^{\hat{\gamma}}x^{\hat{\lambda}} \\ & + \frac{qR_{\hat{\alpha}\hat{\beta}}}{12} \left[\frac{x^{\hat{\alpha}}x^{\hat{\beta}} + 2u^{\hat{\alpha}}x^{\hat{\beta}}(u_{\hat{\gamma}}x^{\hat{\gamma}}) + u^{\hat{\alpha}}u^{\hat{\beta}}(x^{\epsilon}x_{\epsilon} + 2(u_{\hat{\gamma}}x^{\hat{\gamma}})^2)}{\sqrt{q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}}} \pm 2(x^{\hat{\alpha}}u^{\hat{\beta}} + u^{\hat{\alpha}}u^{\hat{\beta}}u_{\hat{\gamma}}x^{\hat{\gamma}}) \right] \\ & - \frac{qR_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}}u^{\hat{\lambda}}u^{\hat{\gamma}}x^{\hat{\alpha}}x^{\hat{\beta}}x^2}{6 (q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}})^{3/2}} \pm q \lim_{h \rightarrow 0} \int_{\mp\infty}^{\tau_{ret/adv} \mp h} G^{ret/adv}(x, z) d\tau + O(\epsilon^2). \quad (39) \end{aligned}$$

Noting that $\hat{S}_0 = r_{\hat{\alpha}}r^{\hat{\alpha}}$, we write Eq. (39) as

$$\begin{aligned} \Phi^{ret/adv} = & \frac{q}{r} \left[1 - \frac{a_{\hat{\gamma}}r^{\hat{\gamma}}x^2}{2r^2} + \frac{3}{8} \left(\frac{a_{\hat{\gamma}}r^{\hat{\gamma}}x^2}{r^2} \right)^2 - \frac{1}{2r^2} \left(\frac{a^2}{12} (r^4 + 6r^2(u_{\hat{\alpha}}x^{\hat{\alpha}})^2 - (u_{\hat{\alpha}}x^{\hat{\alpha}})^4) \right) \right] \\ & - \frac{q}{12r^3} \left[2u_{\hat{\gamma}}x^{\hat{\gamma}}a_{\hat{\mu}}r^{\hat{\mu}}(3r^2 - (u_{\hat{\sigma}}x^{\hat{\sigma}})^2) + 2R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}}x^{\hat{\alpha}}x^{\hat{\beta}}u^{\hat{\gamma}}u^{\hat{\delta}}x^2 - r^2R_{\hat{\alpha}\hat{\beta}}(r^{\hat{\alpha}}r^{\hat{\beta}} + r^2u^{\hat{\alpha}}u^{\hat{\beta}}) \right] \\ & \pm \frac{q}{6} \left[R_{\hat{\alpha}\hat{\beta}}r^{\hat{\alpha}}u^{\hat{\beta}} + 2x^{\hat{\alpha}}(\dot{a}_{\hat{\alpha}} - a^2u_{\hat{\alpha}}) \right] \pm q \lim_{h \rightarrow 0} \int_{\mp\infty}^{\tau_{ret/adv} \mp h} G^{ret/adv}(x, z) d\tau + O(\epsilon^2) \quad (40) \end{aligned}$$

Therefore, using Eq. (15), we can write the gradient of the retarded and advanced fields as

$$\nabla_{\alpha} \Phi^{ret/adv} = \nabla_{\alpha} \left[\left(\frac{qU(x, z)}{\dot{\sigma}} \right)_{ret/adv} \right] - \frac{R(z)q}{12} \left(\frac{\nabla_{\alpha}\hat{S}_0}{2\sqrt{\hat{S}_0}} \pm u_{\alpha} \right) \pm q \nabla_{\alpha} \lim_{h \rightarrow 0} \int_{\mp\infty}^{\tau_{ret/adv} \mp h} G^{ret/adv}(x, z) d\tau. \quad (41)$$

Writing out the gradient of the scalar field in terms of the S_n 's, we have

$$\begin{aligned}
\nabla_{\hat{\alpha}} [\Phi^{ret/adv}] = & q \left[-\frac{\nabla_{\hat{\alpha}} \hat{S}_0}{2\hat{S}_0^{3/2}} - \frac{1}{2} \left(\frac{\nabla_{\hat{\alpha}} \hat{S}_1}{\hat{S}_0^{3/2}} - \frac{3}{2} \frac{\hat{S}_1 \nabla_{\hat{\alpha}} \hat{S}_0}{\hat{S}_0^{5/2}} \right) - \frac{15}{16} \frac{\hat{S}_1^2 \nabla_{\hat{\alpha}} \hat{S}_0}{\hat{S}_0^{7/2}} + \frac{3}{4} \frac{\hat{S}_1 \nabla_{\hat{\alpha}} \hat{S}_1}{\hat{S}_0^{5/2}} \right] \\
& + q \left[-\frac{1}{2} \left(\frac{\nabla_{\hat{\alpha}} \hat{S}_2}{\hat{S}_0^{3/2}} - \frac{3}{2} \frac{\hat{S}_2 \nabla_{\hat{\alpha}} \hat{S}_0}{\hat{S}_0^{5/2}} \right) \pm \frac{1}{6} R_{\hat{\mu}\hat{\beta}} u^{\hat{\beta}} \left(\delta^{\hat{\mu}}_{\hat{\alpha}} + u^{\hat{\mu}} u_{\hat{\alpha}} \right) \right] \\
& + \frac{q R_{\hat{\mu}\hat{\nu}}}{24} \left[\frac{2}{\sqrt{\hat{S}_0}} \left(r^{\hat{\mu}} (\delta^{\hat{\nu}}_{\hat{\alpha}} + u^{\hat{\nu}} u_{\hat{\alpha}}) + u^{\hat{\mu}} u^{\hat{\nu}} \nabla_{\hat{\alpha}} \hat{S}_0 \right) - \frac{\nabla_{\hat{\alpha}} \hat{S}_0}{\hat{S}_0^{3/2}} \left(r^{\hat{\mu}} r^{\hat{\nu}} + \hat{S}_0 u^{\hat{\mu}} u^{\hat{\nu}} \right) \right] \\
& - \frac{q R_{\hat{\mu}\hat{\nu}\hat{\delta}} u^{\hat{\gamma}} u^{\hat{\delta}} x^{\hat{\mu}}}{12 \hat{S}_0^{5/2}} \left(4 \hat{S}_0 x^2 \delta^{\hat{\nu}}_{\hat{\alpha}} + 4 \hat{S}_0 x^{\hat{\nu}} x_{\hat{\alpha}} - 3 x^{\hat{\nu}} x^2 \nabla_{\hat{\alpha}} \hat{S}_0 \right) - \frac{q R(z)}{12} \left(\frac{\nabla_{\hat{\alpha}} \hat{S}_0}{2 \sqrt{\hat{S}_0}} \pm u_{\hat{\alpha}} \right) \\
& \pm q \nabla_{\hat{\alpha}} \lim_{h \rightarrow 0} \int_{\mp \infty}^{\tau_{ret/adv} \mp h} G^{ret/adv}(x, z) d\tau + O(\epsilon^2). \quad (42)
\end{aligned}$$

When re-expressed in terms of a^μ , \dot{a}^μ , and r^μ , $\nabla_{\hat{\alpha}} \Phi^{ret/adv}$ has the form

$$\begin{aligned}
\nabla_{\hat{\alpha}} \Phi^{ret/adv} = & q \left[-\frac{r_{\hat{\alpha}}}{r^3} - \frac{1}{2} \left(\frac{a_{\hat{\alpha}} x^2 + 2 a_{\hat{\gamma}} x^{\hat{\gamma}} x_{\hat{\alpha}}}{r^3} - 3 \frac{a_{\hat{\gamma}} r^{\hat{\gamma}} x^2 r_{\hat{\alpha}}}{r^5} \right) + \frac{3}{4} \frac{a_{\hat{\gamma}} r^{\hat{\gamma}} x^2 (a_{\hat{\alpha}} x^2 + 2 a_{\hat{\gamma}} x^{\hat{\gamma}} x_{\hat{\alpha}})}{r^5} \right] \\
& + q \left[-\frac{15}{8} \frac{(a_{\hat{\gamma}} r^{\hat{\gamma}} x^2)^2 r_{\hat{\alpha}}}{r^7} - \frac{a^2}{24 r^5} \left(r^4 r_{\hat{\alpha}} + 12 r^4 u_{\hat{\gamma}} x^{\hat{\gamma}} u_{\hat{\alpha}} - 6 r^2 (u_{\hat{\gamma}} x^{\hat{\gamma}})^2 r_{\hat{\alpha}} - 4 r^2 (u_{\hat{\gamma}} x^{\hat{\gamma}})^3 u_{\hat{\alpha}} + 3 (u_{\hat{\gamma}} x^{\hat{\gamma}})^4 r_{\hat{\alpha}} \right) \right] \\
& - \frac{q}{2} \left[\left(1 - \frac{1}{3} \left(\frac{u_{\hat{\gamma}} x^{\hat{\gamma}}}{r} \right)^2 \right) \frac{1}{r^3} \left(u_{\hat{\gamma}} x^{\hat{\gamma}} \dot{a}_{\hat{\beta}} x^{\hat{\beta}} r_{\hat{\alpha}} - r^2 (u_{\hat{\alpha}} \dot{a}_{\hat{\beta}} x^{\hat{\beta}} + \dot{a}_{\hat{\alpha}} u_{\hat{\beta}} x^{\hat{\beta}}) \right) + \frac{2 \dot{a}_{\hat{\beta}} x^{\hat{\beta}} (u_{\hat{\gamma}} x^{\hat{\gamma}})^2}{3 r^5} (r^2 u_{\hat{\alpha}} - r_{\hat{\alpha}} u_{\hat{\gamma}} x^{\hat{\gamma}}) \right] \\
& + \frac{q R_{\hat{\mu}\hat{\nu}}}{12} \left[\frac{1}{r} (r^{\hat{\mu}} (\delta^{\hat{\nu}}_{\hat{\alpha}} + u^{\hat{\nu}} u_{\hat{\alpha}}) + 2 u^{\hat{\mu}} u^{\hat{\nu}} r_{\hat{\alpha}}) - \frac{r_{\hat{\alpha}}}{r^3} (r^{\hat{\mu}} r^{\hat{\nu}} + r^2 u^{\hat{\mu}} u^{\hat{\nu}}) \right] \\
& - \frac{q R_{\hat{\mu}\hat{\nu}\hat{\delta}} u^{\hat{\gamma}} u^{\hat{\delta}} x^{\hat{\mu}}}{6 r^5} (2 r^2 x^2 \delta^{\hat{\nu}}_{\hat{\alpha}} + 2 r^2 x^{\hat{\nu}} x_{\hat{\alpha}} - 3 x^{\hat{\nu}} x^2 r_{\hat{\alpha}}) - \frac{q R(z)}{12} \left(\frac{r_{\hat{\alpha}}}{r} \right) \\
& \pm \frac{q}{12} \left[4 (\dot{a}_{\hat{\alpha}} - a^2 u_{\hat{\alpha}}) + 2 R_{\hat{\mu}\hat{\beta}} u^{\hat{\beta}} (\delta^{\hat{\mu}}_{\hat{\alpha}} + u^{\hat{\mu}} u_{\hat{\alpha}}) - R(z) u_{\hat{\alpha}} \right] \pm q \lim_{h \rightarrow 0} \int_{\mp \infty}^{\tau_{ret/adv} \mp h} \nabla_{\hat{\alpha}} G^{ret/adv}(x, z) d\tau + O(\epsilon^2). \quad (43)
\end{aligned}$$

This is a more general expression than is given in Quinn [10]. Only when the field point x is chosen to be along a geodesic orthogonal to the trajectory at $z(0)$ (that is, when $u_{\hat{\alpha}} x^{\hat{\alpha}} = 0$) does this match Quinn's expression.

C. The Singular Field

Using the short-distance expansions we have just presented, we can now identify a singular field, f_{α}^{sing} , satisfying $f_{\alpha}^{ren} = \lim_{x \rightarrow z(0)} [f_{\alpha}^{ret}(x) - f_{\alpha}^{sing}(x)]$, and a singular field, Φ^{sing} , for which

$f_\alpha^{sing} = \nabla^\alpha \Phi^{sing}$. From the explicit form of Eq. (43) for f_α^{ret} we will quickly show that f_α^{sing} comprises all but the last line of Eq. (43) for f_α^{ret} . Then, by recalling the terms in Φ^{ret} that lead to these terms, we write f_α^{sing} in the simpler form,

$$f_\alpha^{sing} = \nabla_\alpha \Phi^{sing}, \quad (44)$$

with

$$\begin{aligned} \Phi^{sing} &= \frac{q}{\sqrt{\hat{S}_0}} - \frac{\hat{S}_1}{2\hat{S}_0^{3/2}} + \left\{ \frac{q}{\sqrt{\hat{S}_0}} \left[\frac{3}{8} \left(\frac{\hat{S}_1}{\hat{S}_0} \right)^2 - \frac{\hat{S}_2^{(1)}}{2\hat{S}_0} \right] - \frac{q}{\sqrt{\hat{S}_0}} \left[\frac{1}{6\hat{S}_0} R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}} u^{\hat{\gamma}} u^{\hat{\delta}} x^{\hat{\alpha}} x^{\hat{\beta}} x^{\hat{\epsilon}} x^{\hat{\epsilon}} \right] \right. \\ &\quad \left. + \frac{q}{\sqrt{\hat{S}_0}} \frac{1}{12} R_{\hat{\alpha}\hat{\beta}} \left[r^{\hat{\alpha}} r^{\hat{\beta}} + u^{\hat{\alpha}} u^{\hat{\beta}} \hat{S}_0 \right] - \frac{1}{12} q R(z) \sqrt{\hat{S}_0} \right\}, \\ &= \Phi^{sing,L} + \Phi^{sing,SL} + \Phi^{sing,SSL}, \end{aligned} \quad (45)$$

where the grouping into three terms exhibits the field as a sum of leading, subleading and sub-subleading terms, of order ϵ^{-1} , ϵ^0 , and ϵ , respectively. Finally, we will check that Eq. (45) is the expansion through $O(\epsilon)$ of the Detweiler-Whiting singular field,

$$\Phi_{DW} = \frac{1}{2} \left[\left(\frac{U(x, z)}{\dot{\sigma}} \right)_{ret} + \left(\frac{U(x, z)}{\dot{\sigma}} \right)_{adv} \right] + \frac{q}{2} \int_{\tau_{ret}}^{\tau_{adv}} V(x, z) d\tau. \quad (46)$$

We begin with the identification of f_α^{sing} with the first five lines of Eq. (43) for $f_\alpha^{ret} = \nabla_{\hat{\alpha}} \Phi^{ret}$. The first three lines of the expression are just $\nabla_{\hat{\alpha}} \tilde{\Phi}$, the gradient of Quinn's flat-space comparison field. The effect of the angle average on the remaining terms is to remove all terms involving the Riemann tensor and its contractions that have an odd number of factors of the coordinates $x^{\hat{\alpha}}$; these are the terms that comprise the 4th and 5th lines of (43). That the angle average of these terms vanishes follows from the fact that the terms are odd under the inversion $I : x^{\hat{\alpha}} \rightarrow -x^{\hat{\alpha}}$, while I maps the domain of integration S_r in the angle average to itself. To make this precise, note that dS is invariant under I and that the restriction to S_r of a function odd under I is a function on S_r that is odd under I . That is to say, $\int_{S_r} f dS = \int_{I(S_r)} f \circ I dS = -\int_{S_r} f dS$, when $f \circ I = -f$. The remaining terms, the last line of Eq. (42), are continuous, and the limit of their angle average is just their value at the particle. Thus, with f_α^{sing} identified with the first five lines of Eq. (43), we have $f_\alpha^{ren} = \lim_{x \rightarrow z(0)} [f_\alpha^{ret}(x) - f_\alpha^{sing}(x)]$, as claimed.

As we have noted, the flat spacetime comparison field $\tilde{\Phi}$ of Eq. (38) includes the leading singular behavior of Φ^{ret} . To obtain the singular field Φ^{sing} , we add the additional $O(\epsilon)$

terms in the short-distance expansion of Φ^{ret} whose gradient provided the terms of f^{sing} involving the Riemann tensor, Ricci tensor, and Ricci scalar terms, the terms in the 4th and 5th lines of (43). The Riemann- and Ricci-tensor terms in $f_{\hat{\alpha}}^{sing}$ are the Riemann- and Ricci-tensor terms of $f_{\hat{\alpha}}^{ret}$ that are odd in $x^{\hat{\alpha}}$, and they are therefore the gradients of the Riemann- and Ricci-tensor terms in the expansion (37) of Φ^{ret} that are even in $x^{\hat{\alpha}}$, namely

$$-\frac{q}{6\hat{S}_0^{3/2}}R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}}u^{\hat{\lambda}}u^{\hat{\gamma}}x^{\hat{\alpha}}x^{\hat{\beta}}x^2 + \frac{qR_{\hat{\alpha}\hat{\beta}}}{12}\left[\frac{r^{\hat{\alpha}}r^{\hat{\beta}} + S_0u^{\hat{\alpha}}u^{\hat{\beta}}}{\sqrt{\hat{S}_0}}\right].$$

Finally, the Ricci scalar term in $f_{\hat{\alpha}}^{sing}$ came from the gradient of the upper limit of integration in the last term of (37): It is the part $\nabla_{\alpha}\tau_{ret}$ that is odd under I . Using Eq. (41), we can write that term as the gradient of

$$-\frac{1}{12}qR\sqrt{\hat{S}_0}.$$

Then

$$\Phi^{sing} = \tilde{\Phi} + -\frac{q}{6\hat{S}_0^{3/2}}R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}}u^{\hat{\lambda}}u^{\hat{\gamma}}x^{\hat{\alpha}}x^{\hat{\beta}}x^2 + \frac{qR_{\hat{\alpha}\hat{\beta}}}{12}\left[\frac{r^{\hat{\alpha}}r^{\hat{\beta}} + S_0u^{\hat{\alpha}}u^{\hat{\beta}}}{\sqrt{\hat{S}_0}}\right] - \frac{1}{12}qR\sqrt{\hat{S}_0}, \quad (47)$$

and Eq. (45) follows. The explicit form of Φ^{sing} in terms of $x^{\hat{\alpha}}$, $u^{\hat{\alpha}}$, and $a^{\hat{\alpha}}$ can be read off from the expanded form (40) Φ^{ret} : The field Φ^{sing} comprises the first two lines of that equation.

From Eq. (43), we can immediately see that, when the test-particle motion is geodesic (when $a^{\alpha} = 0$), Gralla's version of f_{α}^{ren} [26] as an angle average of f_{α}^{ret} holds. Every term in in the first five lines of $f_{\hat{\alpha}}^{sing}$ is then odd under $x^{\hat{\alpha}} \rightarrow -x^{\hat{\alpha}}$, implying $\langle f_{\hat{\alpha}}^{sing} \rangle_r = 0$. We thus have

$$f_{\hat{\alpha}}^{ren} = \lim_{r \rightarrow 0} \langle f_{\hat{\alpha}}^{ret} - f_{\hat{\alpha}}^{sing} \rangle_r = \lim_{r \rightarrow 0} \langle f_{\hat{\alpha}}^{ret} \rangle_r. \quad (48)$$

The singular field Φ^{sing} is not unique. Any function Φ^s defined near the position of the particle is equivalent to Φ^{sing} if it satisfies the conditions

$$\lim_{x \rightarrow z(0)} (\Phi^s - \Phi^{sing}) = 0, \quad \lim_{x \rightarrow z(0)} \nabla_{\alpha} (\Phi^s - \Phi^{sing}) = 0. \quad (49)$$

That is, two singular fields that are equivalent in this sense give the same values of Φ^{ren} and f_{α}^{ren} . We conclude this section by observing that the Detweiler-Whiting singular field, Φ_{DW}^{sing} of Eq. (46), is equivalent in this sense to Φ^{sing} of Eq. (45).

We quickly establish the equivalence as follows. By comparing Eq. (45) for Φ^{sing} to Eq. (37) for $\Phi^{ret/adv}$ we have

$$\Phi^{sing} = \frac{1}{2} (\Phi^{ret} + \Phi^{adv}) - \frac{1}{12} R \sqrt{\hat{S}_0} - \frac{1}{2} q \lim_{h \rightarrow \infty} \left[\int_{-\infty}^{\tau_{ret}-h} G^{ret}(x, z(\tau)) + \int_{\tau_{adv}+h}^{\infty} G^{adv}(x, z(\tau)) \right] d\tau \quad (50)$$

We next use Eq. (14) to write

$$\begin{aligned} \frac{1}{2} (\Phi^{ret} + \Phi^{adv}) &= \frac{1}{2} q \left[\left(\frac{U(x, z)}{\dot{\sigma}} \right)_{ret} + \left(\frac{U(x, z)}{\dot{\sigma}} \right)_{adv} \right] \\ &+ \frac{1}{2} q \lim_{h \rightarrow \infty} \left[\int_{-\infty}^{\tau_{ret}-h} G^{ret}(x, z(\tau)) + \int_{\tau_{adv}+h}^{\infty} G^{adv}(x, z(\tau)) \right]. \end{aligned} \quad (51)$$

From these two equations and the relation $\tau_{adv} - \tau_{ret} = 2\sqrt{\hat{S}_0} + O(\epsilon^2)$, we have the desired result,

$$\begin{aligned} \Phi^{sing} &= \frac{1}{2} q \left[\left(\frac{U(x, z)}{\dot{\sigma}} \right)_{ret} + \left(\frac{U(x, z)}{\dot{\sigma}} \right)_{adv} \right] - \frac{1}{6} q R (\tau_{adv} - \tau_{ret}) + O(\epsilon^2) \\ &= \frac{1}{2} q \left[\left(\frac{U(x, z)}{\dot{\sigma}} \right)_{ret} + \left(\frac{U(x, z)}{\dot{\sigma}} \right)_{adv} \right] + \frac{1}{2} q \int_{\tau_{ret}}^{\tau_{adv}} V d\tau + O(\epsilon^2) \\ &= \Phi_{DW}^{sing} + O(\epsilon^2). \end{aligned} \quad (52)$$

Finally, $\nabla_\alpha \Phi^{sing}$ differs from $\nabla_\alpha \Phi_{DW}^{sing}$ at order ϵ , implying $\lim_{x \rightarrow z(0)} \nabla_\alpha (\Phi_{DW}^{sing} - \Phi^{sing}) = 0$.

III. MODE-SUM REGULARIZATION

We turn now to mode-sum regularization. We extend previous work to include accelerated trajectories on smooth, globally hyperbolic spacetimes with generic smooth coordinate systems, and we show the equivalence of mode-sum regularization to the renormalization methods discussed in the previous section. We begin with a scalar charge and then generalize the results to electromagnetic charges and point masses in the next section.

In mode-sum regularization one writes the retarded and singular fields as sums of angular harmonics, using the fact that the individual harmonics of the retarded field and of the expression for the self-force have finite limits on the particle's trajectory. Because the singular part of the retarded field is defined only in a normal neighborhood of the particle, its individual angular harmonics are defined only after one extends the field to a thick sphere through a position $z(0)$ of the particle. The singular behavior of the retarded field, however, *uniquely determines the large ℓ behavior of its angular harmonics*: For a function f on the

sphere that is smooth everywhere except at a point P , where it has an expansion in powers of the distance to P , that short-distance expansion determines the expansion of the angular harmonics of f in powers of $1/\ell$.

Let (t, r, θ, ϕ) be spherical coordinates related in the usual way to a smooth Cartesian chart (t, x^1, x^2, x^3) for which the 2-spheres of constant t and r are in the domain of the chart. We denote by Φ^{sing} any smooth extension of the singular field of Eq. (45) to a thickened 2-sphere on the $t = 0$ surface through $z(0)$ that includes a finite interval in r about the radial coordinate r_0 of $z(0)$. For Φ representing either Φ^{ret} or Φ^{sing} , each component of the expression for the self-force along the Cartesian coordinate basis has angular harmonics $f_\alpha^{\ell m}$ given by

$$f_\alpha^{\ell m}(t, r) = q \int d\Omega \nabla_\alpha \Phi(t, r, \theta, \phi) \bar{Y}_{\ell m}(\theta, \phi). \quad (53)$$

We have seen that the renormalized self-force at $z(0)$ is given by

$$f_\alpha^{ren} = \lim_{x \rightarrow z(0)} q \nabla_\alpha (\Phi^{ret} - \Phi^{sing}). \quad (54)$$

To obtain an equivalent mode-sum form of f_α^{ren} , we first use the fact that, for $r \neq r_0$ on the thickened sphere where Φ^{sing} is defined, Φ^{ret} and Φ^{sing} are each smooth; second, that their angular harmonics have finite limits as $r \rightarrow r_0^\pm$ (the limits depend whether r approaches r_0 from above or below); and finally that $\nabla_\alpha \Phi^{ret} - \nabla_\alpha \Phi^{sing}$ is continuous on the entire thickened sphere, when its value at $r = r_0$ is taken to be $\lim_{x \rightarrow z(0)} (\nabla_\alpha \Phi^{ret} - \nabla_\alpha \Phi^{sing})$. We then have

$$f_\alpha^{ren}/q = \lim_{r \rightarrow r_0} \nabla_\alpha (\Phi^{ret} - \Phi^{sing})(t = 0, r, \theta_0, \phi_0) \quad (55)$$

$$= \lim_{r \rightarrow r_0} \sum_{\ell, m} [\nabla_\alpha (\Phi^{ret} - \Phi^{sing})]^{\ell m}(t = 0, r) Y_{\ell m}(\theta_0, \phi_0) \quad (56)$$

$$= \sum_{\ell, m} \lim_{r \rightarrow r_0} [\nabla_\alpha (\Phi^{ret} - \Phi^{sing})]^{\ell m}(t = 0, r) Y_{\ell m}(\theta_0, \phi_0) \quad (57)$$

$$= \sum_{\ell, m} \left[\lim_{r \rightarrow r_0^\pm} (\nabla_\alpha \Phi^{ret})^{\ell m}(t = 0, r) - \lim_{r \rightarrow r_0^\pm} (\nabla_\alpha \Phi^{sing})^{\ell m}(t = 0, r) \right] Y_{\ell m}(\theta_0, \phi_0), \quad (58)$$

where r_0 , θ_0 , and ϕ_0 are the angular coordinates of the particle at time $t = 0$.

The finite range of the sum over m allows the definitions

$$f_{\alpha}^{ret,\ell\pm} := q \sum_{m=-\ell}^{\ell} \lim_{r \rightarrow r_0^{\pm}} \nabla_{\alpha} \Phi^{ret,\ell m}(t=0, r) Y_{\ell m}(\theta_0, \phi_0), \quad (59a)$$

$$f_{\alpha}^{sing,\ell\pm} := q \sum_{m=-\ell}^{\ell} \lim_{r \rightarrow r_0^{\pm}} \nabla_{\alpha} \Phi^{sing,\ell m}(t=0, r) Y_{\ell m}(\theta_0, \phi_0), \quad (59b)$$

and the renormalized self-force is then given by

$$f_{\alpha}^{ren} = \sum_{\ell=0}^{\infty} f_{\alpha}^{ren,\ell} := \sum_{\ell=0}^{\infty} (f_{\alpha}^{ret,\ell\pm} - f_{\alpha}^{sing,\ell\pm}). \quad (60)$$

We will show that $f_{\alpha}^{sing,\ell\pm}$ has the form,

$$f_{\alpha}^{sing,\ell\pm} = \pm A_{\alpha} L + B_{\alpha}. \quad (61)$$

where $L \equiv \ell + 1/2$, and A_{α} and B_{α} are constants independent of ℓ . This form was obtained for a scalar charge moving on a geodesic in a Kerr background by Barack and Ori [2] and for a massive particle (in a Lorenz gauge) by Barack [3]. We show here that the form is valid in our more general context of accelerated motion in a smooth, globally hyperbolic spacetime.

Roughly speaking, functions g on the sphere that diverge as $1/\theta^k$ near $\theta = 0$ have angular harmonics g^{ℓ} for which $\sum_{\ell=0}^{\ell_{max}} g^{\ell}$ diverges as ℓ_{max}^k .⁶ For an expansion in ϵ whose leading term is ϵ^{-2} , one then anticipates a singular field for which

$$f_{\alpha}^{sing,\ell} = A_{\alpha} L + B_{\alpha} + C_{\alpha} L^{-1} + O(L^{-2}), \quad (62)$$

again with A_{α} , B_{α} , and C_{α} constants independent of ℓ . The leading and subleading terms, $A_{\alpha} L$ and B_{α} , arise from the $1/\epsilon^2$ (Coulomb) behavior of f_{α}^{ret} . A term C_{α}/L would yield a logarithmic divergence in the sum

$$\sum_{\ell=0}^{\ell_{max}} C_{\alpha}/L = C_{\alpha} \log \ell_{max} + O(\ell_{max}^{-1});$$

because this would correspond to a (nonexistent) $\log \epsilon$ term in the short-distance expansion of f_{α}^{ret} , it cannot be present. The argument can be made precise:⁷ After subtracting the

⁶ Functions of this kind belong to Sobolev spaces H_s with $s < 0$, and the relation between the singular behavior of functions in H_s and that of their angular harmonics is described in Appendix B of [27].

⁷ This was pointed out to us by Sam Gralla

leading and subleading terms from the singular field, the remainder is defined and uniformly bounded everywhere on the sphere except at a point (the position of the particle), where it is direction-dependent. Its angular transform is therefore convergent, implying that no term of the form $1/L$ can be present. Our calculation in Sec. III A below explicitly verifies that $C_\alpha = 0$.

Finally, terms of order ϵ^0 in f_α^{sing} (terms of order L^{-2} or higher, including terms falling off faster than any power of L) could in principle contribute a finite term Δ_α ,

$$\Delta_\alpha = \sum_{\ell=0}^{\infty} (f_\alpha^{sing,\ell} - A_\alpha L + B_\alpha). \quad (63)$$

Following [1], we refer to $A_\alpha, B_\alpha, C_\alpha$ and Δ_α as ‘regularization parameters’.⁸ The goals of this section are to show that Δ_α vanishes and finding the explicit form of A_α and B_α .

Because Eqs. (59) involve sums over all m , the values of $f_\alpha^{ret,\ell\pm}$ and $f_\alpha^{sing,\ell\pm}$ are invariant under a rotation of the (θ, ϕ) coordinates. To evaluate them, it is convenient to choose rotated coordinates (that we again denote by θ, ϕ) for which the particle is on the coordinate axis, $\theta = 0$ at $z(0)$. Using $Y_{\ell m}(\theta = 0, \phi) = 0 \ \forall \ m \neq 0$ and Eqs. (53) and (59b), we can write

$$f_\alpha^{sing,\ell\pm} \equiv [\nabla_\alpha \Phi^{sing}]^\ell = \lim_{r \rightarrow r_0^\pm} \frac{L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \Phi^{sing}. \quad (64)$$

To calculate the regularization parameters, we use Eq. (64) with Φ^{sing} given by the expression in Eq. (45). The singular field is expressed in terms of RNCs, but the integral in Eq. (64) is over a sphere that can be arbitrarily large. We therefore need to extend the singular field to the entire sphere. As mentioned above, two extensions that differ by a smooth function with support outside a neighborhood of the particle do not alter the singular field.

Because the mode-sum involves spherical harmonics associated with a specified coordinate system (t, r, θ, ϕ) , we begin by rewriting the short-distance expansion Eq. (45) as an expansion in terms of the coordinate distances to the particle. To do so, we define Cartesian coordinates x^μ (termed “locally Cartesian angular coordinates” in [1]) associated with these coordinate differences by

$$x^0 = t, \quad x^1 = x = \rho(\theta) \cos \phi, \quad x^2 = y = \rho(\theta) \sin \phi, \quad x^3 = r - r_0, \quad (65)$$

⁸ In [1], the term Δ_α is written as D_α . Other authors, however, reserve the symbol D_α for the coefficient of L^{-2} in the expansion of the singular field (see, for example [28]). We introduce Δ_α to avoid confusion.

where $\rho(\theta) = 2 \sin(\theta/2)$. In choosing these coordinates – in particular, choosing $\rho(\theta)$ instead of $\sin \theta$ – and in subsequently discarding terms of order ϵ^2 , we need to check that different choices give the same angular harmonic series up to convergent terms whose sum vanishes at the particle. We can see that this is the case, because two choices of $\rho(\theta)$ that differ by terms of order θ^3 and for which the corresponding values of $\nabla \rho$ differ by $O(\theta^2)$ give expansions of each component $\nabla_\alpha \Phi^{sing}$ that differ by a continuous function that is $O(\epsilon)$. The difference in the angular harmonic series of each component $\nabla_\alpha \Phi^{sing}$ is therefore a series that converges to zero at the particle. The values of the regularization parameters A_α and B_α , regarded as vectors, depend on the original coordinate system (t, r, θ, ϕ) , but not on the locally Cartesian coordinates we use to evaluate them. Their components, of course, depend on the choice of basis.

The coordinates x^μ are related to RNCs $x^{\hat{\alpha}}$ by

$$x^{\hat{\alpha}} = \partial_\mu x^{\hat{\alpha}} x^\mu + \frac{1}{2} \partial_\mu x^{\hat{\alpha}} \Gamma^\mu_{\epsilon\nu} x^\epsilon x^\nu + \frac{1}{6} \partial_\mu x^{\hat{\alpha}} (\Gamma^\mu_{\nu\gamma} \Gamma^\gamma_{\epsilon\lambda} + \partial_\lambda \Gamma^\mu_{\nu\epsilon}) x^\epsilon x^\nu x^\lambda + \dots \quad (66)$$

When we use this relation to replace the RNCs by the coordinates x^μ , the expansion Eq. (45) retains the same form, with \hat{S}_0 , \hat{S}_1 , and \hat{S}_2 replaced by quantities S_0 , S_1 , and S_2 , where

$$S_0 := q_{\mu\nu} x^\mu x^\nu, \quad (67)$$

$$S_1 := \left(a_\lambda g_{\mu\nu} + \frac{1}{2} g_{\mu\nu, \lambda} + u_\epsilon u_\lambda \Gamma^\epsilon_{\mu\nu} \right) x^\mu x^\nu x^\lambda =: 2\zeta_{\mu\nu\lambda} x^\mu x^\nu x^\lambda, \quad (68)$$

with all quantities in parentheses evaluated at $z(0)$. We will not use the explicit expression for S_2 and do not give it here because of its length; we need only the fact that it is a homogeneous polynomial of degree 4 in the coordinates x^μ .

From Eq. (45), the singular field's leading order term is $O(\epsilon^{-1})$, and the leading-order term in its derivative is $O(\epsilon^{-2})$. Recalling Eq. (64), we write

$$f_\alpha^{sing, \ell} = f_\alpha^{L, \ell} + f_\alpha^{SL, \ell} + f_\alpha^{SSL, \ell}, \quad (69)$$

where $f_\alpha^{L, \ell}$, $f_\alpha^{SL, \ell}$, and $f_\alpha^{SSL, \ell}$ denote respectively the contributions to f_α^{sing} at leading, sub-leading, and sub-subleading order. From Eq. (45), they are given by the following expres-

sions, evaluated on the $t = 0$ surface:

$$f_{\alpha}^{L,\ell} = \lim_{r \rightarrow r_0^{\pm}} q \int d\Omega P_{\ell}(\cos(\theta)) \nabla_{\alpha} \Phi^{sing,L}, \quad (70a)$$

$$f_{\alpha}^{SL,\ell} = \lim_{r \rightarrow r_0^{\pm}} q \int d\Omega P_{\ell}(\cos(\theta)) \nabla_{\alpha} \Phi^{sing,SL}, \quad (70b)$$

$$f_{\alpha}^{SSL,\ell} = \lim_{r \rightarrow r_0^{\pm}} q \int d\Omega P_{\ell}(\cos(\theta)) \nabla_{\alpha} \Phi^{sing,SSL}. \quad (70c)$$

In the remainder of this section, we use Eq. (70), with $\Phi^{sing,L}$, $\Phi^{sing,SL}$, and $\Phi^{sing,SSL}$ given by Eq. (45), to show that the large ℓ behavior of f_{α}^{sing} given in Eq. (61) follows from the general character of the short-distance form of Φ^{sing} , given in Eqs. (71) below. We then find the explicit forms of A_{α} and B_{α} . Denoting by $P^{(k)}(x^{\mu})$ a homogeneous polynomial of degree k in the coordinates x^{μ} , we write the leading, subleading, and sub-subleading terms of Φ^{sing} in the form

$$\Phi^{sing,L} = \frac{C}{\hat{S}_0^{1/2}} \quad (71a)$$

$$\Phi^{sing,SL} = \frac{P^{(3)}(x^{\mu})}{\hat{S}_0^{3/2}} \quad (71b)$$

$$\Phi^{sing,SSL} = \frac{P^{(6)}(x^{\mu})}{\hat{S}_0^{5/2}}. \quad (71c)$$

For $\Phi^{sing,L}$ and $\Phi^{sing,SL}$, this form is explicit in Eq. (45); for $\Phi^{sing,SSL}$, terms are grouped with the common denominator $\hat{S}_0^{5/2}$.

That the mode-sum expression (61) holds for electromagnetic and gravitational perturbations will again follow from the fact that each component of the corresponding singular fields (the singular parts of the perturbed vector potential and metric) satisfies Eq. (71).

Our treatment of Eqs. (70b) and (70c) differs from that of Eq. (70a). In the former cases, we are allowed to take the limit inside the integral, which simplifies the calculation. In the latter case we cannot do this. The fact that the limit and integral commute follows from the fact that, after one writes $d\Omega = d\theta d\phi \sin\theta$, the integrands in Eqs. (70b) and (70c) are bounded functions of θ and ϕ and are defined everywhere except at $\theta = 0$.⁹ We examine

⁹ The result is an immediate consequence of Lebesgue's dominated convergence theorem (see, for example, [29], p. 191): Let $\{F_n\}$ be a sequence of integrable functions that converges almost everywhere to F . If $|F| < G$, for some integrable function G , then F is integrable and $\int F d\mu = \lim_{n \rightarrow \infty} \int F_n d\mu$. For functions of the type we consider here, a proof can also be found in [1].

these subleading and sub-subleading terms before evaluating the leading term.

Thus far, we have been following the methods of Barack and Ori [1] exactly. At this point they used properties of the Schwarzschild geometry, and we rephrase the argument in a way that holds for a general background spacetime.

A. The Sub-Sub-Leading term

The sub-subleading term in the self-force is the easiest to evaluate, and we will see that it vanishes. A function $\Phi^{sing,SSL}$ of the form (71c) has gradient of the form

$$\nabla_\alpha \Phi^{sing,SSL} = \frac{P_\alpha^{(7)}(x^\mu)}{S_0^{7/2}}, \quad (72)$$

where each component $P_\alpha^{(7)}$ is a homogeneous polynomial of degree 7. Because only polynomials in the three coordinates x^i , $i = 1, \dots, 3$ survive when $f_\alpha^{SSL,\ell}$ is evaluated on the $t = 0$ surface, we have

$$f_\alpha^{SSL,\ell} = \lim_{r \rightarrow r_0} \frac{q^2 L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \frac{P_\alpha^{(7)}(x^i)}{S_0^{7/2}}. \quad (73)$$

That a function of the form $P^{(k)}(x^i)/S_0^{k/2}$ is bounded follows immediately from the definition (67) of S_0 and the fact that the spatial part q_{ij} of $q_{\mu\nu}$ is positive definite. As noted above, we can then interchange the order of the limit and integration. To see that the integral over the sphere at $r = r_0$ vanishes, we use the fact that $P^{(7)}$ is odd under $I : x^\mu \rightarrow -x^\mu$, while S_0 is even (see the specific discussion in next section, after Eq. (77)). From Eq. (65) the restriction of I to the $t = 0$, $r = r_0$ sphere is the map $\phi \rightarrow \phi + \pi$, implying that the sphere itself and the measure $d\Omega$ are invariant under I . Because the integrand is odd under I and $d\Omega$ is invariant, the integral vanishes.

B. The Subleading Term

The subleading term of Eq. (70b),

$$f_\alpha^{SL,\ell} = \lim_{r \rightarrow r_0^\pm} \frac{q^2 L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \left(-\frac{S_1}{2S_0^{3/2}} \right), \quad (74)$$

is more singular than the sub-subleading term by an additional power of $S_0^{1/2}$ in its denominator. It has the form

$$f_\alpha^{SL,\ell} = \lim_{r \rightarrow r_0^\pm} \frac{q^2 L}{2\pi} \int d\theta d\phi \sin \theta P_\ell(\cos \theta) \frac{P_\alpha^{(2n)}(x^i)}{S_0^{n+1/2}}, \quad (75)$$

To compute the explicit form of $f_\alpha^{SL,\ell}$ and to see that $\sin \theta \frac{P_\alpha^{(2n)}(x^i)}{S_0^{n+1/2}}$ is bounded, we begin by noting that, restricted to the $r = r_0, t = 0$ sphere, $P_\alpha^{(2n)}$ and S_0 are given by

$$P_\alpha^{(2n)}(x^i)|_{r=r_0} = \rho(\theta)^{2n} \left(\sum_{m=0}^{2n} a_{\alpha,m} \sin^m \phi \cos^{2n-m} \phi \right), \quad (76)$$

where $a_{\alpha,m}$ is a constant; and

$$\tilde{S}_0 := S_0|_{r=r_0} = \rho(\theta)^2 (q_{xx} \cos(\phi)^2 + q_{yy} \sin(\phi)^2), \quad (77)$$

where we have used the fact that, with our rotated θ, ϕ coordinates, $q_{xy} = 0$. In effect, this is exactly what Barack and Ori [1] do for Schwarzschild, choosing their coordinates such that $u_y = 0$, and then relying on the diagonal form of the metric to make $q_{xy} = 0$. Then, because the eigenvalues of q_{IJ} , $I, J = 1 \dots 2$, are positive definite, S_0 can be written as

$$\tilde{S}_0 = \rho(\theta)^2 q_{yy} (1 + \beta^2 \cos^2 \phi), \quad (78)$$

where

$$\beta^2 := \frac{q_{xx} - q_{yy}}{q_{yy}}. \quad (79)$$

From Eqs. (76) and (78), it follows that $S_0^{n+1/2}$ has one more power of $\rho(\theta)$ than $P_\alpha^{(2n)}$ and hence that the integrand, $\sin \theta P_\ell(\cos \theta) P_\alpha^{(2n)} S_0^{-(n+1/2)}$, is bounded.

We can therefore again bring the limit inside the integral in Eq. (75). Substituting the expressions (76) and (78) for $P_\alpha^{(2n)}$ and \tilde{S}_0 in Eq. (75), we have

$$f_\alpha^{SL,\ell} = \frac{q^2 L}{2\pi q_{yy}^{n-1/2}} \int_0^\pi d\theta \sin \theta \frac{P_\ell(\cos(\theta))}{\rho(\theta)} \sum_{m=0}^{2n} \int_0^{2\pi} \frac{(a_{\alpha,m} \sin^m \phi \cos^{2n-1-m} \phi)}{(1 + \beta^2 \cos^2 \phi)^{(2n-1)/2}} d\phi. \quad (80)$$

The integral over θ has the value

$$\int_0^\pi d\theta \sin \theta \frac{P_\ell(\cos(\theta))}{\rho(\theta)} = \int_0^\pi d\theta \sin \theta \frac{P_\ell(\cos(\theta))}{\sqrt{2 - 2 \cos(\theta)}} = \frac{1}{L}, \quad (81)$$

implying $f_\alpha^{SL,\ell}$ is independent of ℓ :

$$f_\alpha^{SL,\ell} = B_\alpha. \quad (82)$$

The integration over ϕ involves the complete elliptic integrals

$$E(w) = \int_0^{\pi/2} (1 - w \sin^2 \phi)^{1/2} d\phi, \quad K(w) = \int_0^{\pi/2} (1 - w \sin^2 \phi)^{-1/2} d\phi, \quad (83)$$

where

$$w := \frac{\beta^2}{1 + \beta^2}. \quad (84)$$

After a straightforward computation, we find

$$B_\alpha = \frac{2q^2}{3\pi(1 + \beta^2)^{3/2}\beta^4 q_{yy}^{5/2}} (B_\alpha^{(E)} E(w) + B_\alpha^{(K)} K(w)), \quad (85)$$

where

$$B_\alpha^{(E)} = (1 + \beta^2)(2 + \beta^2)\Lambda_{\alpha XXY Y} - 2[(1 + 2\beta^2)\Lambda_{\alpha xxx x} + (1 + \beta^2)^2(1 - \beta^2)\Lambda_{\alpha yyy y}] \quad (86a)$$

$$B_\alpha^{(K)} = (2 + 3\beta^2)\Lambda_{\alpha xxx x} + (1 + \beta^2)[(2 - \beta^2)\Lambda_{\alpha yyy y} - 2\Lambda_{\alpha XXY Y}], \quad (86b)$$

with the quantities $\Lambda_{\alpha\beta\gamma\delta\epsilon}$ given in terms of $\zeta_{\beta\gamma\delta}$ of Eq. (68) by

$$\Lambda_{\alpha\beta\gamma\delta\epsilon} := 3\zeta_{(\alpha\beta\gamma)}q_{\delta\epsilon} - 3\zeta_{\beta\gamma\delta}q_{\alpha\epsilon}, \quad (87)$$

and we define the $\Lambda_{\alpha XXY Y}$ as follows;

$$\Lambda_{\alpha XXY Y} = \Lambda_{\alpha xxyy} + \Lambda_{\alpha xyxy} + \Lambda_{\alpha yyyx} + x \leftrightarrow y. \quad (88)$$

In summary, we have shown that the angular harmonic decomposition of the subleading term has only a B term, a term independent of ℓ , whose explicit form is given by Eqs. (85)-(87).

These parameters agree with those of Barack and Ori for Schwarzschild [1], and also with Warburton and Barack [30] and [31] in Kerr. (In particular note the equivalence of our Eq. (87) with Eqs. (B5), (B6) and (B7) of [30]).

C. Leading Term

Finally, we turn to the leading term $f_\alpha^{L,\ell}$. From Eq. (70a) and the relation $\nabla_\alpha S_0 = 2q_{\alpha\beta}x^\beta$, we have

$$f_{\alpha\pm}^{L,\ell} = -\frac{L}{2\pi} q^2 q_{\alpha\beta} \tilde{F}_\pm^{\beta\ell}, \quad (89)$$

where

$$\tilde{F}_{\pm}^{\beta\ell} = \lim_{r \rightarrow r_0^{\pm}} \int d\Omega P_{\ell}(\cos(\theta)) \frac{x^{\beta}}{S_0^{3/2}}. \quad (90)$$

Because we are working on a $t = 0$ surface, we have $\tilde{F}_{\pm}^{0\ell} = 0$. To evaluate $\tilde{F}_{\pm}^{i\ell}$, we follow Barack and Ori [1], dividing the $r = \text{constant}$ sphere that constitutes the domain of integration into two parts: the coordinate square \mathcal{S}_{ϵ} for which $|x| < \epsilon$ and $|y| < \epsilon$ (some $\epsilon < \pi/2$); and the rest of the sphere, $S^2 \setminus \mathcal{S}_{\epsilon}$. The domains are chosen to be symmetric under a rotation by π about $\theta = 0$.

On $S^2 \setminus \mathcal{S}_{\epsilon}$, the integrand is smooth, and we can bring the limit inside the integral, writing

$$\lim_{r \rightarrow r_0^{\pm}} \int_{S^2 \setminus \mathcal{S}_{\epsilon}} d\Omega P_{\ell}(\cos(\theta)) \frac{x^i}{S_0^{3/2}} = \int_{S^2 \setminus \mathcal{S}_{\epsilon}} d\Omega P_{\ell}(\cos(\theta)) \frac{x^i}{\tilde{S}_0^{3/2}}.$$

We immediately see that the contribution to the radial component $\tilde{F}_{\pm}^{1\ell}$ vanishes. The remaining x and y components of the integral vanish because the domain of integration and the function \tilde{S}_0 are invariant under a rotation by π about $\theta = 0$, while x and y change sign.

The only contribution to $\tilde{F}_{\pm}^{\beta\ell}$ is then from the integral over \mathcal{S}_{ϵ} . Because ϵ is arbitrary, the value of the integral is independent of ϵ , determined only by the singular behavior of the integrand at $\theta = 0$. To evaluate the integral, we change integration variables from (θ, ϕ) to (x, y) . From Eq. (65), the Jacobian of the transformation is

$$\frac{\partial(\theta, \phi)}{\partial(x, y)} = \sin \theta, \quad (91)$$

and we have

$$\tilde{F}_{\pm}^{i\ell} = \lim_{r \rightarrow r_0^{\pm}} \int_{\mathcal{S}_{\epsilon}} dx dy P_{\ell}(\cos \theta) \frac{x^i}{S_0^{3/2}} = \lim_{r \rightarrow r_0^{\pm}} \int_{-\epsilon}^{\epsilon} dx \int_{-\epsilon}^{\epsilon} dy P_{\ell}(\cos \theta) \frac{x^i}{S_0^{3/2}}. \quad (92)$$

Because $P_{\ell}(\cos \theta)$ differs from its value at $\theta = 0$ only at $O(\theta^2)$, replacing P_{ℓ} by 1 does not alter the leading singular behavior of the integrand and should therefore not change the value of the integral. To verify this, we write

$$P_{\ell}(\cos \theta) = 1 + h(\theta) \sin^2 \theta, \quad (93)$$

where h is smooth on \mathcal{S}_{ϵ} . We then have

$$\tilde{F}_{\pm}^{i\ell} = \lim_{r \rightarrow r_0^{\pm}} \int_{\mathcal{S}_{\epsilon}} dx dy \frac{x^i}{S_0^{3/2}} + \int_{\mathcal{S}_{\epsilon}} dx dy \lim_{r \rightarrow r_0^{\pm}} \left(h \sin^2 \theta \frac{x^i}{S_0^{3/2}} \right) \equiv \left(\lim_{r \rightarrow r_0^{\pm}} I_1^i \right) + I_2^i,$$

where we have used the fact that the function $h \sin^2 \theta x^i / S_0^{3/2}$ is bounded to bring the limit inside the second integral, I_2^i . Then I_2^i has the form

$$I_2^i = \int_{\mathcal{S}_\epsilon} dx dy \left(h \sin^2 \theta \frac{x^i}{\tilde{S}_0^{3/2}} \right). \quad (94)$$

Again the vanishing of I_2^r is immediate, and the symmetry argument we have now used twice implies that the remaining components also vanish: That is, from the invariance of \mathcal{S}_ϵ and $h \sin^2 \theta / \tilde{S}_0^{3/2}$ under a π rotation, together with the fact that x and y change sign, we have $I_2^x = I_2^y = 0$.

We are now left with

$$\tilde{F}_\pm^{i\ell} = \lim_{r \rightarrow r_0^\pm} \int_{\mathcal{S}_\epsilon} \frac{x^i}{S_0^{3/2}} dx dy. \quad (95)$$

We can already see that this integral is independent of L , because P_ℓ has been replaced by 1. It immediately follows from Eq. (89) that $f_\alpha^{L,\ell}$ is proportional to L , and *we have thus established our central claim, that the singular part of the self-force has the form given in Eq. (1).*

Finally, we evaluate $\tilde{F}_\pm^{i\ell}$ to find the explicit form of A_α . We begin by showing that the x - and y -components can be expressed in terms of the third spatial component $\tilde{F}_\pm^{r\ell}$. From the definition (67) of S_0 , we have

$$\partial_x \frac{1}{S_0^{1/2}} = - \frac{q_{xx}x + q_{xr}(r - r_0)}{S_0^{3/2}}, \quad (96)$$

and the x -component of Eq. (95) takes the form

$$\tilde{F}_\pm^{x\ell} = - \frac{1}{q_{xx}} \lim_{r \rightarrow r_0^\pm} \int_{\mathcal{S}_\epsilon} \left[\partial_x \frac{1}{S_0^{1/2}} + \frac{q_{xr}}{S_0^{3/2}} (r - r_0) \right] dx dy. \quad (97)$$

Using $\int_{-\epsilon}^\epsilon dx \partial_x S_0^{-1/2} = 0$, we have

$$\tilde{F}_\pm^{x\ell} = - \frac{q_{xr}}{q_{xx}} \lim_{r \rightarrow r_0^\pm} \int_{\mathcal{S}_\epsilon} \frac{r - r_0}{S_0^{3/2}} dx dy = - \frac{q_{xr}}{q_{xx}} \tilde{F}_\pm^{r\ell}, \quad (98)$$

as claimed. Similarly,

$$\tilde{F}_\pm^{y\ell} = - \frac{q_{yr}}{q_{yy}} \tilde{F}_\pm^{r\ell}. \quad (99)$$

To evaluate $\tilde{F}_\pm^{r\ell}$, we introduce as integration variables

$$X = \frac{x}{r - r_0}, \quad Y = \frac{y}{r - r_0}. \quad (100)$$

With $e : \epsilon/(r - r_0)$, we have

$$\begin{aligned}\tilde{F}_\pm^{r\ell} &= \lim_{e \rightarrow \infty} \int_{-e}^e dX \int_{-e}^e dY [q_{xx}X^2 + 2q_{xr}X + q_{yy}Y^2 + 2q_{yr}Y + q_{rr}]^{-3/2} \\ &= \pm 2\pi (q_{xx}q_{yy}q_{rr} - q_{yy}q_{xr}^2 - q_{xx}q_{yr}^2)^{-1/2}.\end{aligned}\tag{101}$$

Finally, using $f_{\alpha\pm}^{L,\ell} = A_\alpha L$, together with Eqs. (89), (98), (99) and (101), we obtain

$$A_{\alpha\pm} = \mp q^2 \frac{q_{\alpha r} - q_{\alpha x}q_{xr}/q_{xx} - q_{\alpha y}q_{yr}/q_{yy}}{(q_{xx}q_{yy}q_{rr} - q_{yy}q_{xr}^2 - q_{xx}q_{yr}^2)^{1/2}}.\tag{102}$$

It is worth noting that this agrees with the form given in [1] and also has the same property that $u_\alpha A^\alpha = 0$.

Thus, as claimed, the regularization parameters for the self force on a point scalar charge moving along an arbitrary trajectory through a generic spacetime are given by $A_\alpha L + B_\alpha$, with the terms for a logarithmic divergence ($C_\alpha L^{-1}$) and a finite remainder (Δ_α) both vanishing. We have given the explicit forms of the regularization parameters in the ‘locally Cartesian angular coordinates,’ in Eqs. (102) and (85). Their values for the original coordinate system are given in Appendix C.

It is important to note that we have recovered the regularization parameters for $f_\alpha^{sing,\ell}$, whose values are not (necessarily) trivially related to those for $f^{sing,\ell,\alpha}$. For now we will just claim that the parameters for the raised indices, the regularization parameters have the form, $A^\alpha L + B^\alpha$, and postpone the proof to the end of next section, where we can discuss it in the context of extending the four velocity away from the world-line.

We now turn to the regularization parameters for electromagnetism and gravity.

IV. GENERALIZING TO HIGHER SPINS

In this section, to distinguish an electromagnetic vector potential from the regularization parameter A^α , we use a different font, denoting the vector potential by \mathbf{A}^α .

We will see that, in a Lorenz gauge, each Cartesian component of the vector potential \mathbf{A}^α of an electric point charge and of the metric perturbation $h_{\alpha\beta}$ of a point mass has a short-distance expansion similar to that of the field of a scalar charge. We show that this similarity of form implies that the mode-sum expression for the singular part of the retarded field is again given by Eq. (61). In particular the term Δ_α again vanishes. We again rely on the Hadamard expansion of the Green’s functions as laid out in [7].

A. Electromagnetic Self-Force

In a Lorenz gauge, the electromagnetic vector potential \mathbf{A}^α of a point charge e satisfies

$$\nabla^\beta \nabla_\beta \mathbf{A}^\alpha - R^\alpha{}_\beta \mathbf{A}^\beta = -4\pi j^\alpha, \quad \nabla_\alpha \mathbf{A}^\alpha = 0, \quad (103)$$

with current density

$$j^\alpha(x) = eu^\alpha(x) \int \delta^{(4)}(x, z(\tau)) d\tau. \quad (104)$$

The solution to Eq. (103) has components \mathbf{A}^μ in a global coordinate system given by

$$\mathbf{A}_{adv/ret}^\mu(x) = \int [G_{\nu'}^\mu(x, x')]_{adv/ret} j^{\nu'}(x') \sqrt{-g} d^4 x', \quad (105)$$

where each Green's function satisfies the equation

$$\nabla^\gamma \nabla_\gamma G_{\hat{\beta}}^\alpha(x, x') - R^\alpha{}_\beta G_{\hat{\beta}}^\beta(x, x') = -4\pi \delta_{\hat{\beta}}^\alpha \delta^{(4)}(x, x'). \quad (106)$$

Unprimed and primed indices are tensor indices at x and x' , respectively, and the covariant derivatives are with respect to x .

The expansion of the Green's function in the normal neighborhood C is analogous to that of the scalar field, having the form [7]

$$G_{\beta'}^\alpha(x, x') = \Theta(x, x') [U_{\beta'}^\alpha(x, x') \delta(\sigma) - V_{\beta'}^\alpha(x, x') \theta(-\sigma)], \quad (107)$$

where the bi-tensors $U_{\beta'}^\alpha(x, x')$ and $V_{\beta'}^\alpha(x, x')$ have in RNC the local expansions

$$U_{\hat{\beta}}^{\hat{\alpha}}(x, x') = \delta_{\hat{\beta}}^{\hat{\alpha}} + \frac{1}{12} [2R_{\hat{\gamma}\hat{\beta}\hat{\delta}}^{\hat{\alpha}} + \delta_{\hat{\beta}}^{\hat{\alpha}} R_{\hat{\gamma}\hat{\delta}}] y^{\hat{\gamma}} y^{\hat{\delta}} + O(\epsilon^3) \quad (108)$$

and

$$V_{\hat{\beta}}^{\hat{\alpha}} = \frac{1}{2} \left(R_{\hat{\beta}}^{\hat{\alpha}} - \frac{1}{6} \delta_{\hat{\beta}}^{\hat{\alpha}} R \right) + O(\epsilon). \quad (109)$$

In these expansions, each tensor is evaluated at the point x' .

The same steps we followed for the scalar field now give for each component of \mathbf{A}^α essentially the same form as that of the scalar field in Eq. (14), namely

$$\mathbf{A}_{adv/ret}^\alpha = e \left[\frac{U_{\beta'}^\alpha u^{\beta'}}{\dot{\sigma}} \right]_{adv/ret} \mp e \lim_{h \rightarrow 0^+} \int_{\pm\infty}^{\tau_{adv/ret} \pm h} u^{\nu'} [G_{\nu'}^\alpha]_{adv/ret} d\tau. \quad (110)$$

The force has the formal expression

$$f_{EM}^\alpha = -\nabla_\beta T_{EM}^{\alpha\beta} = F^{\alpha\beta} j_\beta, \quad (111)$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, and the expression for the singular part of the force is given in terms of the singular part of the vector potential by

$$f_{EM}^{sing,\alpha} = eu^\beta g^{\alpha\sigma} [\nabla_\sigma A_\beta^{sing} - \nabla_\beta A_\sigma^{sing}] , \quad (112)$$

where components of the metric and 4-velocity are evaluated at the position of the particle.

B. Gravitational Self-Force

The test-particle limit of the trajectory of a massive particle moving in a curved spacetime is a geodesic. To consistently compute the self-force on a massive particle whose trajectory is accelerated in the test-particle limit, one must include whatever additional fields are responsible for the acceleration. In this section, we find the formal contribution from gravity to the self-force on a particle in a generic vacuum spacetime, showing that the form (61) holds in this general context.

As noted in the introduction, we find that the expression can be used in a mode-sum regularization of a particle with scalar or electromagnetic charge moving in a spacetime with a background scalar or electromagnetic field, when one works to linear order in the mass and charge (with a fixed ratio q/m). That is, we assume that renormalization can again be accomplished by an angle average of the retarded field together with a mass renormalization. From a short-distance expansion of the retarded field, we then obtain the singular field and find that the regularization coefficients have the form $A_\alpha L + B_\alpha$ with A_α the sum of the coefficients for gravity and electromagnetism obtained here, and with B_α having an additional contribution from the coupling of the two fields.

Returning to the task of this section, finding the regularization parameters for gravity, we will write the spacetime metric as $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$, where $\tilde{g}_{\alpha\beta}$ is the total metric, $g_{\alpha\beta}$ is the background metric, and $h_{\alpha\beta}$ is the perturbation. We will restrict our discussion to background metrics $g_{\alpha\beta}$ that satisfy the vacuum Einstein equation. We raise and lower indices with the background metric $g_{\alpha\beta}$ and denote by ∇_α the covariant derivative operator of $g_{\alpha\beta}$.

With $\gamma_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h$, the Lorenz gauge condition is $\nabla_\alpha \gamma^{\alpha\beta} = 0$, and the linearized Einstein equation has the form

$$\nabla_\mu \nabla^\mu \gamma^{\alpha\beta} + 2R_\gamma{}^\alpha{}_\delta{}^\beta \gamma^{\gamma\delta} = -16\pi T^{\alpha\beta}. \quad (113)$$

Here, $T^{\alpha\beta}$ is the stress energy tensor of a point particle of mass m , given by

$$T^{\alpha\beta} = mu^\alpha u^\beta \int \delta^{(4)}(x' - z(\tau)) d\tau. \quad (114)$$

As before, we write the solution to the field equation (in this case, Eq. (113)) in terms of a Green's function,

$$\gamma^{\alpha\beta} = 4 \int G^{\alpha\beta}_{\gamma'\delta'}(x, x') T^{\gamma'\delta'} \sqrt{-g'} d^4 x', \quad (115)$$

where $G^{\alpha\beta}_{\gamma'\delta'}(x, x')$ satisfies

$$\nabla_\mu \nabla^\mu G^{\alpha\beta}_{\gamma'\delta'}(x, x') + 2R_{\gamma}^{\alpha}{}_{\delta}{}^{\beta} G^{\gamma\delta}_{\gamma'\delta'}(x, x') = -4\pi g^{(\alpha}_{\gamma'} g^{\beta)}_{\delta'} \delta^4(x, x'). \quad (116)$$

As in the spin-0 and spin-1 cases, the Green's function, $G^{\alpha\beta}_{\gamma'\delta'}(x, x')$, has the form

$$G^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}}(x, x') = \Theta(x, x') \left[U^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}}(x, x') \delta(\sigma) - V^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}}(x, x') \theta(-\sigma) \right], \quad (117)$$

where the bitensors $U^{\alpha\beta}_{\gamma'\delta'}$ and $V^{\alpha\beta}_{\gamma'\delta'}$ have, in RNC about x , the expansions [7]

$$U^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}}(x, x') = \delta_{\hat{\gamma}}^{(\hat{\alpha}} \delta_{\hat{\delta}}^{\hat{\beta})} + \frac{1}{3} \delta_{(\hat{\gamma}}^{(\hat{\alpha}} R_{\hat{\delta})\hat{\sigma}}^{\hat{\beta})} x^{\hat{\sigma}} x^{\hat{\mu}} + O(\epsilon^3), \quad (118a)$$

$$V^{\hat{\alpha}\hat{\beta}}_{\hat{\gamma}\hat{\delta}}(x, x') = R_{\hat{\gamma}}^{(\hat{\alpha}\hat{\beta})}{}_{\hat{\delta}} + O(\epsilon). \quad (118b)$$

When we evaluate the perturbation using Eq. (115), we find

$$\gamma^{\alpha\beta}_{adv/ret} = 4m \left[\frac{u^{\gamma'} u^{\delta'} U^{\alpha\beta}_{\gamma'\delta'}}{\dot{\sigma}} \right]_{adv/ret} \mp 4m \lim_{h \rightarrow 0^+} \int_{\pm\infty}^{\tau_{adv/ret} \pm h} u^{\gamma'} u^{\delta'} \left[G^{\alpha\beta}_{\gamma'\delta'} \right]_{adv/ret} d\tau. \quad (119)$$

Now, solving the perturbed geodesic equation allows us to write

$$f_{GR}^{\alpha, sing} = -m (g^{\alpha\delta} + u^\alpha u^\delta) \left(\nabla_\beta h_{\gamma\delta}^{sing} - \frac{1}{2} \nabla_\delta h_{\beta\gamma}^{sing} \right) u^\beta u^\gamma. \quad (120)$$

Therefore, just as for the scalar charge in Eqs. (14) and (15), and as for the electric charge in Eqs. (110) and (112), we have expressed the metric perturbation in Eq. (119) and the expression for the force in Eq. (120).

C. The vanishing C and Δ terms

We will now argue that the C_α and Δ_α terms vanish in both the electrodynamic and gravitational self-force regularization (computed in a Lorenz gauge). In the scalar case, the singular field was given by Eq. (52) reproduced here

$$\Phi^{sing} = \frac{1}{2} q \left[\left(\frac{U(x, z)}{\dot{\sigma}} \right)_{ret} + \left(\frac{U(x, z)}{\dot{\sigma}} \right)_{adv} \right] + q \frac{1}{2} \int_{\tau_{ret}}^{\tau_{adv}} V d\tau + O(\epsilon^2).$$

To produce the singular vector potential A^α and metric perturbation, $\gamma^{\alpha\beta}$, we will use the same procedure, taking the averaged sum of the advanced and retarded solutions of the function “ U ” (Eqs. (108) and (118a)) over $\dot{\sigma}$, and add to it the leading order contribution from the function “ V ” (Eqs. (109) and (118b)).

Now, before we continue, it is useful to recall the properties of Φ^{sing} which we noted gave rise to the vanishing C_α and Δ_α terms for a scalar charge. In particular,

$$\Phi^{sing} = \frac{1}{\sqrt{S_0}} + \frac{\zeta_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma}{S_0^{3/2}} + \frac{\Lambda_{\alpha\beta\gamma\delta\epsilon\sigma} x^\alpha x^\beta x^\gamma x^\delta x^\epsilon x^\sigma}{S_0^{5/2}} + O(\epsilon^2).$$

The A_α term came from the gradient of the leading term, and the B_α term came from the gradient of the sub-leading term. When we considered the mode-sum decomposition of the sub-sub-leading term, we noted that its contribution vanished because, to borrow the expression from Quinn, it had an odd number of unit normal vectors.

Therefore, our aim will be to demonstrate that both A_{sing}^α and $\gamma_{sing}^{\alpha\beta}$ (and therefore also $h_{sing}^{\alpha\beta}$) have the same form as Φ^{sing} , and that this will lead to $C_\alpha = \Delta_\alpha = 0$.

By comparing with Eq. (52), we notice that the only term from the integral over “ V ” will be the leading order term in the expansion multiplied by $\sqrt{S_0}$. That is, the Detweiler-Whiting piece for electromagnetism and gravity will just take on the form of $\sqrt{S_0} \times Const$. The gradient of this term will just give an odd polynomial in x divided by $\sqrt{S_0}$, and so its mode-sum decomposition vanishes.

Having established this, we can focus on the direct piece of the singular field. The only qualitatively new feature that arises in the direct part of the field is the presence of the four velocity in the numerator. Consider the explicit expression for the four velocity at the retarded or advanced times:

$$u_{ret/adv}^\alpha = u^\alpha + a^\alpha(\tau_1 + \tau_2 + \dots) + \frac{1}{2}\dot{a}^\alpha(\tau_1 + \tau_2 + \dots)^2 + \dots \quad (121)$$

By using Eqs. (25) and (26), we can rewrite $u_{ret/adv}^\alpha$ in terms of the coordinates of x as

$$\begin{aligned} u_{ret/adv}^{\hat{\alpha}} &= u^{\hat{\alpha}} - a^{\hat{\alpha}} u_{\hat{\mu}} x^{\hat{\mu}} + \left[a^{\hat{\alpha}} a_{\hat{\mu}} u_{\hat{\nu}} + \frac{1}{2} \dot{a}^{\hat{\alpha}} (q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}} u_{\hat{\nu}}) \right] x^{\hat{\mu}} x^{\hat{\nu}} \\ &\pm \left[\frac{x^{\hat{\gamma}}}{2} (a^{\hat{\alpha}} a_{\hat{\gamma}} (q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}} u_{\hat{\nu}}) + 2 \dot{a}^{\hat{\alpha}} q_{\hat{\mu}\hat{\nu}} u_{\hat{\gamma}}) - a^{\hat{\alpha}} q_{\hat{\mu}\hat{\nu}} \right] \frac{x^{\hat{\mu}} x^{\hat{\nu}}}{\sqrt{\hat{S}_0}} \end{aligned} \quad (122)$$

Now, if we turn to U^α_β in Eq. (108), and we note that to leading order $y^\alpha = x^\alpha - u^\alpha \tau_1$, we can write

$$U^{\hat{\alpha}}_{\hat{\beta}} = \delta^{\hat{\alpha}}_{\hat{\beta}} + \frac{(-2R^{\hat{\alpha}}_{(\hat{\gamma}\hat{\delta})\hat{\beta}} + R_{\hat{\gamma}\hat{\delta}}\delta^{\hat{\alpha}}_{\hat{\beta}})}{12} \left[\delta^{\hat{\gamma}}_{\hat{\mu}}\delta^{\hat{\delta}}_{\hat{\nu}} + u^{\hat{\gamma}}u^{\hat{\delta}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}}) + 2u^{\hat{\gamma}}\delta^{\hat{\delta}}_{\hat{\nu}}u_{\hat{\mu}} \right] x^{\hat{\mu}}x^{\hat{\nu}} \\ \pm \frac{u^{\hat{\gamma}}(-2R^{\hat{\alpha}}_{(\hat{\gamma}\hat{\delta})\hat{\beta}} + R_{\hat{\gamma}\hat{\delta}}\delta^{\hat{\alpha}}_{\hat{\beta}})}{6} \left[u^{\hat{\delta}}u_{\hat{\nu}}x^{\hat{\nu}} + x^{\hat{\delta}} \right] \sqrt{S_0} \quad (123)$$

Using Eqs. (122) and (123), we obtain

$$\left[U^{\hat{\alpha}}_{\hat{\beta}} u^{\hat{\beta}} \right]_{ret/adv} = u^{\hat{\alpha}} - a^{\hat{\alpha}}u_{\hat{\gamma}}x^{\hat{\gamma}} + \left[a^{\hat{\alpha}}u_{\hat{\mu}}a_{\hat{\nu}} + \frac{\dot{a}^{\hat{\alpha}}}{2}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}}) \right] x^{\hat{\mu}}x^{\hat{\nu}} + \\ \frac{(u^{\hat{\alpha}}R_{\hat{\gamma}\hat{\delta}} - 2R^{\hat{\alpha}}_{(\hat{\gamma}\hat{\delta})\hat{\beta}}u^{\hat{\beta}})}{12} \left[\delta^{\hat{\gamma}}_{\hat{\mu}}\delta^{\hat{\delta}}_{\hat{\nu}} + u^{\hat{\gamma}}u^{\hat{\delta}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}}) + 2u^{\hat{\gamma}}\delta^{\hat{\delta}}_{\hat{\nu}}u_{\hat{\mu}} \right] x^{\hat{\mu}}x^{\hat{\nu}} \\ \pm \frac{x^{\hat{\mu}}x^{\hat{\nu}}}{\sqrt{S_0}} \left[-a^{\hat{\alpha}}q_{\hat{\mu}\hat{\nu}} + \frac{x^{\hat{\gamma}}}{6} \left(3a^{\hat{\alpha}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})a_{\hat{\gamma}} + 6\dot{a}^{\hat{\alpha}}u_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}} + (u^{\hat{\alpha}}R_{\hat{\epsilon}\hat{\delta}}u^{\hat{\delta}} - 2R^{\hat{\alpha}}_{(\hat{\epsilon}\hat{\delta})\hat{\beta}}u^{\hat{\beta}})q^{\hat{\epsilon}}_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}} \right) \right] \quad (124)$$

Now, recalling Eq. (29) we can write the direct piece of the electromagnetic vector potential,

$$\left[\frac{U^{\hat{\alpha}}_{\hat{\beta}} u^{\hat{\beta}}}{\dot{\sigma}} \right] = \frac{u^{\hat{\alpha}}}{\sqrt{S_0}} \left[1 - \frac{S_1}{2S_0} + \frac{3S_1^2}{8S_0^2} - \frac{S_2^{(1)}}{2S_0} - \frac{R_{\hat{\mu}\hat{\nu}\hat{\epsilon}\hat{\delta}}x^{\hat{\mu}}u^{\hat{\nu}}x^{\hat{\epsilon}}u^{\hat{\delta}}x^2}{6S_0} \right] - \frac{a^{\hat{\alpha}}u_{\hat{\mu}}x^{\hat{\mu}}}{\sqrt{S_0}} \left(1 - \frac{S_1}{2S_0} \right) \\ + \frac{[2a^{\hat{\alpha}}u_{\hat{\mu}}a_{\hat{\nu}} + \dot{a}^{\hat{\alpha}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})]x^{\hat{\mu}}x^{\hat{\nu}}}{2\sqrt{S_0}} \\ \frac{(u^{\hat{\alpha}}R_{\hat{\gamma}\hat{\delta}} - 2u^{\hat{\beta}}R^{\hat{\alpha}}_{(\hat{\gamma}\hat{\delta})\hat{\beta}})}{12\sqrt{S_0}} \left[\delta^{\hat{\gamma}}_{\hat{\mu}}\delta^{\hat{\delta}}_{\hat{\nu}} + u^{\hat{\gamma}}u^{\hat{\delta}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}}) + 2u^{\hat{\gamma}}\delta^{\hat{\delta}}_{\hat{\nu}}u_{\hat{\mu}} \right] x^{\hat{\mu}}x^{\hat{\nu}} \pm \frac{a^{\hat{\alpha}}S_1}{2S_0} \mp u^{\hat{\alpha}}\frac{S_2^{(\pm)}}{2S_0^{3/2}} \\ \pm \frac{x^{\hat{\mu}}x^{\hat{\nu}}}{S_0} \left[-a^{\hat{\alpha}}q_{\hat{\mu}\hat{\nu}} + \frac{x^{\hat{\gamma}}}{6} \left(3a^{\hat{\alpha}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})a_{\hat{\gamma}} + 6\dot{a}^{\hat{\alpha}}u_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}} + (u^{\hat{\alpha}}R_{\hat{\epsilon}\hat{\delta}}u^{\hat{\delta}} - 2R^{\hat{\alpha}}_{(\hat{\epsilon}\hat{\delta})\hat{\beta}}u^{\hat{\beta}})q^{\hat{\epsilon}}_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}} \right) \right], \quad (125)$$

where we have decomposed S_2 into two pieces, $S_2^{(1)}$, which does not change sign when switching from retarded to advanced times, and $S_2^{(\pm)}$, which does.

In the average of the retarded and advanced fields, the contribution from each term in

Eq. (125) preceded by \pm vanishes, so we can write the singular vector potential as,

$$\begin{aligned} \frac{1}{e} \mathbf{A}_{sing}^{\hat{\alpha}} = & \frac{u^{\hat{\alpha}}}{\sqrt{S_0}} \left[1 - \frac{S_1}{2S_0} + \frac{3S_1^2}{8S_0^2} - \frac{S_2^{(1)}}{2S_0} - \frac{R_{\hat{\mu}\hat{\nu}\hat{\epsilon}\hat{\delta}} x^{\hat{\mu}} u^{\hat{\nu}} x^{\hat{\epsilon}} u^{\hat{\delta}} x^2}{6S_0} \right] - \frac{a^{\hat{\alpha}} u_{\hat{\mu}} x^{\hat{\mu}}}{\sqrt{S_0}} \left(1 - \frac{S_1}{2S_0} \right) \\ & + \frac{(u^{\hat{\alpha}} R_{\hat{\gamma}\hat{\delta}} - 2u^{\hat{\beta}} R_{(\hat{\gamma}\hat{\delta})\hat{\beta}}^{\hat{\alpha}})}{12\sqrt{S_0}} \left[\delta_{\hat{\mu}}^{\hat{\gamma}} \delta_{\hat{\nu}}^{\hat{\delta}} + u^{\hat{\gamma}} u^{\hat{\delta}} (q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}} u_{\hat{\nu}}) + 2u^{\hat{\gamma}} \delta_{\hat{\nu}}^{\hat{\delta}} u_{\hat{\mu}} \right] x^{\hat{\mu}} x^{\hat{\nu}} \\ & + \frac{[2a^{\hat{\alpha}} u_{\hat{\mu}} a_{\hat{\nu}} + \dot{a}^{\hat{\alpha}} (q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}} u_{\hat{\nu}})] x^{\hat{\mu}} x^{\hat{\nu}}}{2\sqrt{S_0}} + \frac{6R_{\hat{\beta}}^{\hat{\alpha}} u^{\hat{\beta}} - u^{\hat{\alpha}} R}{12} \sqrt{S_0}. \end{aligned} \quad (126)$$

The last term is the ‘Detweiler and Whiting term’ which is $V(0)\sqrt{S_0}$. The first term of Eq. (126) is just u^{α} multiplied by the scalar field. The second term contains a sub-leading contribution, which is of the form of a linear term divided by $\sqrt{S_0}$. As we saw in the previous section, terms of this type provide a ‘B’ term. All of the rest of the terms (the final term from the first line, the total of the second line, and the beginning of the third) are terms of order ϵ , and have the form of a polynomial of even degree in x divided by S_0 raised to a half integer power. Therefore, the derivatives of these terms will give us polynomials of odd integer powers in the numerator, and thus their mode-sum decomposition will vanish.

Therefore, the regularization parameters for the electrodynamic self-force will also be of the form $A^{\alpha}L + B^{\alpha}$. It is also worth noting that the regularization parameters for the electrodynamic vector potential will just be of the form B'^{α} (we use B' to indicate that this will not have the same value as the B for the self-force).

When we apply the same procedure to Eq. (118a) and solve for the retarded and advanced $\gamma_{\alpha\beta}$, we find

$$\begin{aligned} \frac{1}{m} \gamma_{\hat{\alpha}\hat{\beta}}^{ret/adv} = & \frac{4u_{\hat{\alpha}} u_{\hat{\beta}}}{\sqrt{S_0}} \left[1 - \frac{S_1}{2S_0} + \frac{3S_1^2}{8S_0^2} - \frac{S_2^{(1)}}{2S_0} - \frac{R_{\hat{\mu}\hat{\nu}\hat{\epsilon}\hat{\delta}} x^{\hat{\mu}} u^{\hat{\nu}} x^{\hat{\epsilon}} u^{\hat{\delta}} x^2}{6S_0} \right] - 8 \frac{u_{(\hat{\beta}} a_{\hat{\alpha})} u_{\hat{\mu}} x^{\hat{\mu}}}{\sqrt{S_0}} \left(1 - \frac{S_1}{2S_0} \right) \\ & + \frac{4x^{\hat{\mu}} x^{\hat{\nu}}}{\sqrt{S_0}} \left[(a_{\hat{\alpha}} a_{\hat{\beta}} + \dot{a}_{(\hat{\alpha}} u_{\hat{\beta})}) (q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}} u_{\hat{\nu}}) + 2a_{(\hat{\alpha}} u_{\hat{\beta}}) a_{\hat{\mu}} u_{\hat{\nu}} - \frac{u_{(\hat{\alpha}} R_{\hat{\beta})\hat{\epsilon}\hat{\delta}} u^{\hat{\delta}}}{3} (\delta_{\hat{\mu}}^{\hat{\epsilon}} \delta_{\hat{\nu}}^{\hat{\delta}} + u^{\hat{\epsilon}} \delta_{\hat{\mu}}^{\hat{\delta}} u_{\hat{\nu}}) \right] \\ & \pm 8u_{(\hat{\alpha}} a_{\hat{\beta})} \left(1 - \frac{S_1}{2S_0} \right) \pm \frac{8x^{\hat{\mu}} x^{\hat{\nu}} x^{\hat{\delta}}}{S_0} \left[(a_{\hat{\alpha}} a_{\hat{\beta}} + \dot{a}_{(\hat{\alpha}} u_{\hat{\beta})}) u_{\hat{\delta}} q_{\hat{\mu}\hat{\nu}} - a_{(\hat{\alpha}} u_{\hat{\beta}}) a_{\hat{\delta}} (q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}} u_{\hat{\nu}}) \right] \mp 2 \frac{u_{\hat{\alpha}} u_{\hat{\beta}} S_2^{(\pm)}}{S_0^{3/2}} \\ & - 4u^{\hat{\mu}} u^{\hat{\nu}} R_{\hat{\mu}(\hat{\alpha}\hat{\beta})\hat{\nu}} (\sqrt{S_0} \mp u_{\hat{\mu}} x^{\hat{\mu}}) \end{aligned} \quad (127)$$

Therefore, we can write the singular, trace-reversed, metric perturbation as

$$\begin{aligned} \frac{1}{m}\gamma_{\hat{\alpha}\hat{\beta}}^{sing} = & \frac{4u_{\hat{\alpha}}u_{\hat{\beta}}}{\sqrt{S_0}} \left[1 - \frac{S_1}{2S_0} + \frac{3S_1^2}{8S_0^2} - \frac{S_2^{(1)}}{2S_0} - \frac{R_{\hat{\mu}\hat{\nu}\hat{\epsilon}\hat{\sigma}}x^{\hat{\mu}}u^{\hat{\nu}}x^{\hat{\epsilon}}u^{\hat{\sigma}}x^2}{6S_0} \right] - 8\frac{u_{(\hat{\beta}}a_{\hat{\alpha})}u_{\hat{\mu}}x^{\hat{\mu}}}{\sqrt{S_0}} \left(1 - \frac{S_1}{2S_0} \right) + \\ & \frac{4x^{\hat{\mu}}x^{\hat{\nu}}}{\sqrt{S_0}} \left[(a_{\hat{\alpha}}a_{\hat{\beta}} + \dot{a}_{(\hat{\alpha}}u_{\hat{\beta})})(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}}) + 2a_{(\hat{\alpha}}u_{\hat{\beta}})a_{\hat{\mu}}u_{\hat{\nu}} - \frac{u_{(\hat{\alpha}}R_{\hat{\beta})\hat{\epsilon}\gamma\hat{\sigma}}u^{\gamma}}{3}(\delta_{\hat{\mu}}^{\hat{\epsilon}}\delta_{\hat{\nu}}^{\hat{\sigma}} + u^{\hat{\epsilon}}\delta_{\hat{\mu}}^{\hat{\sigma}}u_{\hat{\nu}}) \right] \\ & - 4u^{\hat{\mu}}u^{\hat{\nu}}R_{\hat{\mu}(\hat{\alpha}\hat{\beta})\hat{\nu}}\sqrt{S_0}. \end{aligned} \quad (128)$$

When we write $h_{\alpha\beta}^{sing} = \gamma_{\alpha\beta} - 1/2g_{\mu\nu}\gamma_{\mu}^{\mu}$, we will only introduce one new term, which will be from $R_{\alpha\gamma\beta\delta}x^{\gamma}x^{\delta}(S_0^{-1/2})$ (from the Riemann tensor correction to the metric multiplied by the leading order term in γ_{μ}^{μ}). This is an order ϵ term already in the form of an even polynomial divided by S_0 to a half integer power. Therefore, looking at the singular field in Eq. (128), we can see that it also has the exact same algebraic form as the scalar and electrodynamic singular fields. Therefore, the mode-sum decomposition of the derivative of this field will have the form $A_{...}L + B_{...}$ (where the ‘...’ represent suppressed indices). However, for the gravitational self-force, we also have terms that are linear in the field, not just the derivatives (since $\nabla \rightarrow \partial + \Gamma$).

When we expand the Cristoffel symbols, then we will get $\Gamma \rightarrow \Gamma_0 + \partial_{\mu}\Gamma x^{\mu} + O(\epsilon^2)$. When we apply this to the singular metric perturbation, (recalling that the sub-sub-leading terms vanish, since they are already of order ϵ^1), we will have something of the form

$$\Gamma_{...}\gamma_{...} = \frac{C_{...}^{(1)}}{\sqrt{S_0}} + \frac{C_{...}^2x^{\mu}x^{\nu}x^{\delta}}{S_0^{3/2}} + O(\epsilon).$$

The $C_{...}^{(n)}$ are constants. The mode-sum decomposition of the first term gives us a piece independent of L (another ‘ B ’ term), and the second term is an odd polynomial divided by S_0 to an odd integer power, and so this term’s mode-sum decomposition vanishes.

The expressions for the self-force in an electromagnetic or gravitational context depend on how one extends $g^{\alpha\beta}[z(0)]$ and $u^{\alpha}[z(0)]$ to a neighborhood of the particle (and there is even this ambiguity in how one defines the scalar self-force with raised indices). If we return to the definition of the scalar, electromagnetic, or gravitational self-force, (Eqs. (54), (112)

or (120), then we can rewrite them as

$$\begin{aligned}
f^{s=0,sing,\mu} &= k^{\mu\nu} \nabla_\nu \Phi^{sing} = g^{\mu\nu} \nabla_\nu \Phi^{sing} \\
f^{s=1,sing,\mu} &= k^{\mu\alpha\beta} \nabla_\beta \mathbf{A}_\alpha^{sing} = (\delta^{\mu\beta} u^\alpha - \delta^{\mu\alpha} u^\beta) \nabla_\beta \mathbf{A}_\alpha^{sing} \\
f^{s=2,sing,\mu} &= k^{\mu\beta\gamma\delta} \nabla_\beta \gamma_{\gamma\delta}^{sing} = \frac{(q^{\beta\mu} (q^{\gamma\delta} + u^\gamma u^\delta) - 4q^{\delta\mu} u^\beta u^\gamma)}{4} \nabla_\beta \gamma_{\gamma\delta}^{sing}.
\end{aligned} \tag{129}$$

In particular, the quantities $k^{\mu\dots}$ are only properly defined on the trajectory of the particle for $s = 1, 2$, and we are allowed a choice in how we extend $k^{\mu\dots}$ away from the world line. One popular way is to use the ‘fixed extension’ [3], in which one defines $k^{\mu\dots}(x \neq z(0)) = k^{\mu\dots}(x = z(0))$, and is the one we use in this paper, but other choices are available [24]. We now show that as long as $k^{\mu\dots}$ is a smooth function in x then the regularization parameters retain the form $A_\alpha L + B_\alpha$.

Since each component of \mathbf{A}_α^{sing} and $\gamma_{\alpha\beta}^{sing}$ has the same form as Φ^{sing} , we will consider finding the regularization parameters for $f^{s=0,sing,\mu}$. Denote by $k_0^{\mu\nu}$, $\partial_\gamma k_0^{\mu\nu}$, and $\partial_\delta \partial_\gamma k_0^{\mu\nu}$ the values of $k^{\mu\nu}$ and its derivatives at $z(0)$. For an extension $k^{\mu\nu}[x]$ of $k^{\mu\nu}[z(0)]$ the departure of $k^{\mu\nu} \nabla_\nu \Phi^{sing}$ from $k_0^{\mu\nu} \nabla_\nu \Phi^{sing}$ is given by

$$\begin{aligned}
(k^{\mu\nu} - k_0^{\mu\nu}) \nabla_\nu \Phi^{sing} &= x^\gamma \partial_k^{\mu\nu} g_0 \nabla_\nu \phi^{sing,L} + \left(x^\gamma \partial_\gamma k_0^{\mu\nu} \nabla_\nu \phi^{sing,SL} + \frac{1}{2} x^\gamma x^\delta \partial_\gamma \partial_\delta k^{\mu\nu} \nabla_\nu \phi^{sing,L} \right) \\
&+ O(\epsilon).
\end{aligned} \tag{130}$$

The first term on the right has the form $P^{(4)}(x^\mu) S_0^{-5/2}$, and it thus gives a correction to the B term. The term in parentheses on the right is order unity and has the form $P^{(7)}(x^\mu) S_0^{-7/2}$; its contribution to the $f^{SSL,\ell}$, given by its contribution to the integral on the right side of Eq. (73) therefore vanishes. Because the remaining part of the right side of (130) is $O(\epsilon)$, its contribution to the f_α^{sing} also vanishes.

Therefore, we have demonstrated our claim in Eq. (1). In doing so, we have shown that to regularize the fields themselves, one needs only subtract of a ‘ B ’ term from the mode-sum of the retarded field, which is to say, for a field ψ , $\psi_{\dots}^{sing,\ell} = B_{\dots}$. We give the explicit values of the self-force regularization parameters in Appendix A, and the expressions for these parameters in the original coordinate system in Appendix C.

It is important to note that even though the higher order terms in the expansion of the singular field do not contribute to the entire mode-sum, they do contribute mode by mode. That is to say, that when we perform the infinite sum over all modes, the higher order terms

vanish, but they do not vanish mode by mode. As we noted in a previous footnote, these terms are important for increasing the speed of convergence in computations [28].

V. THE RADIAL SELF-FORCE ON A STATIC SCALAR CHARGE OUTSIDE OF A SCHWARZSCHILD BLACK HOLE

We will write the Schwarzschild metric in the usual manner,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2, \quad (131)$$

where M is the mass of the black hole. We will now proceed to calculate the regularization parameters for a static charge outside this black hole.

Given a static charge outside a Schwarzschild black hole, we can place the particle at the pole, allowing us to write, $z^\alpha(0) = (0, r_0, 0, 0)$. From the normalization of the four-velocity, $u_\mu u^\mu = -1$ we can write the four velocity as

$$u_\mu = -\left(\sqrt{1 - \frac{2M}{r_0}}, 0, 0, 0\right). \quad (132)$$

The four acceleration is given by $a^\mu \equiv u^\nu \nabla_\nu u^\mu$, allowing us to write

$$a_\mu = \left(0, \frac{M}{r_0^2 - 2Mr_0}, 0, 0\right). \quad (133)$$

To use the results of Eqs. (85) and (102), we need to write the line element in terms of our LCACs. This is trivially done for the static charge in Schwarzschild, for which $g_{\delta t \delta t} = g_{tt}$, $g_{\delta r \delta r} = g_{rr}$, and $g_{xx} = g_{yy} = r_0^2$.

With this information we can use Eq. (102) to write

$$A_{r\pm}L = \mp \frac{Lq^2}{r_0^2 \sqrt{1 - \frac{2M}{r_0}}}. \quad (134)$$

To calculate the B_r term, we can use Eq. (85), but, since $\beta^2 \equiv g_{yy}^{-1}(g_{xx} - g_{yy} + u_x^2) = 0$ it is actually easier to return to the integral in Eq. (70b) reproduced below,

$$B_\alpha = \lim_{\delta r \rightarrow 0^\pm} \frac{q^2 L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \left(\frac{3}{4} \frac{S_1 \nabla_\alpha S_0}{S_0^{5/2}} - \frac{1}{2} \frac{\nabla_\alpha S_1}{S_0^{3/2}} \right). \quad (135)$$

From Eqs. (67), (68), (132), and (133), we have

$$\begin{aligned} S_0 &= \frac{\delta r^2}{1 - \frac{2M}{r_0}} + r_0^2 \rho(\theta)^2 \\ S_1 &= r_0 \delta r \rho(\theta)^2 \left[1 + \frac{M}{r_0} \frac{1}{1 - \frac{2M}{r_0}} \right]. \end{aligned} \quad (136)$$

Since we can bring the limit inside the integral in Eq. (135), B_t , B_x . and B_y all vanish, and B_r has the form

$$B_r = -\frac{q^2}{2r_0^2} \left[1 + \frac{M}{r_0} \frac{1}{1 - \frac{2M}{r_0}} \right]. \quad (137)$$

Eqs. (134) and (137) match the corresponding Eqs. (10.17a) and (10.17b) in Casals et. al. [32].

A. The Retarded Field

From Wiseman [15], the retarded field has the form

$$\phi_{ret} = q \frac{\sqrt{1 - \frac{2M}{r_0}}}{\sqrt{(r - M)^2 - 2(r - M)(r_0 - M) \cos(\theta) + (r_0 - M)^2 - M^2 \sin^2(\theta)}}, \quad (138)$$

Using the relation

$$\frac{1}{\sqrt{a^2 + b^2 - 2abx - (1 - x^2)}} = \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(x) P_{\ell}(a) Q_{\ell}(b), \quad (139)$$

we can write Eq. (138) as

$$\phi_{ret} = \frac{q}{M} \sqrt{1 - \frac{2M}{r_0}} \sum_{\ell=0}^{\infty} 2L P_{\ell}(\cos(\theta)) P_{\ell}\left(\frac{r_{<}}{M} - 1\right) Q_{\ell}\left(\frac{r_{>}}{M} - 1\right), \quad (140)$$

where Q_{ℓ} is a Legendre function of the second kind, $r_{>} = \max(r, r_0)$, and $r_{<} = \min(r, r_0)$.

Now, to make the comparison of the mode sum decomposition of the retarded field and of the singular field easier, we define

$$\begin{aligned} F_{+, \ell} &:= \left[\frac{q}{2} (\partial_{r_{>}} + \partial_{r_{<}}) \phi_{ret} \right]_{r=r_0} \\ F_{-, \ell} &:= \left[\frac{q}{2} (\partial_{r_{>}} - \partial_{r_{<}}) \phi_{ret} \right]_{r=r_0} \end{aligned} \quad (141)$$

Thus, if we want to know the force from the retarded field for $r > r_0$, then we can write $F_{>, \ell} = F_{+, \ell} + F_{-, \ell}$. Similarly $F_{<, \ell} = F_{+, \ell} - F_{-, \ell}$. Notice that this suggests we can write

$$\begin{aligned} F_{+, \ell} &= B + F_{\ell}^{(self)} \\ F_{-, \ell} &= A_{+} L. \end{aligned} \quad (142)$$

The anti-symmetric term is the easier one to calculate. Using the formula for the Wronskian of P_{ℓ} and Q_{ℓ} given in Eq. (8.1.9) in Abromowitz and Stegun [33], we can write

$$F_{-, \ell} = -\frac{q^2 L}{r_0^2 \sqrt{1 - \frac{2M}{r_0}}} = A_{+} L. \quad (143)$$

Thus, we have seen that our formula in Eq. (102) has successfully found the leading order term for the static charge in Schwarzschild.

Now, we can write the symmetric term, $F_{+, \ell}$ as

$$F_{+, \ell} = \frac{q^2}{2M} \sqrt{\frac{b-M}{b+M}} \partial_b \left[(2\ell+1) P_\ell \left(\frac{b}{M} \right) Q_\ell \left(\frac{b}{M} \right) \right], \quad (144)$$

where $b \equiv r_0 - M$. Now, we will rewrite the B_r term given in Eq. (137) in a similar form, yielding

$$B_r = \frac{q^2}{2} \sqrt{\frac{b-M}{b+M}} \partial_b \left[\frac{1}{\sqrt{b^2 - M^2}} \right]. \quad (145)$$

In the limit that $\ell_{max} \rightarrow \infty$, we can write

$$f_r^{self} = \sum_{\ell=0}^{\ell_{max}} [(F_{+, \ell} + F_{-, \ell}) - (A_+ L + B_r)] = \sum_{\ell=0}^{\ell_{max}} [(-F_{+, \ell} + F_{-, \ell}) - (A_- L + B_r)]. \quad (146)$$

Since we showed that the $F_{-, \ell}$ term is exactly canceled by the A term, we can just focus on the difference between the $F_{+, \ell}$ and the B_r term. From Wiseman's result, [15], we know that the self-force vanishes. Therefore we need to show that,

$$\frac{q^2}{2} \sqrt{\frac{b-M}{b+M}} \partial_b \sum_{\ell=0}^{\infty} \left[\frac{(2\ell+1)}{M} P_\ell \left(\frac{b}{M} \right) Q_\ell \left(\frac{b}{M} \right) - \frac{1}{\sqrt{b^2 - M^2}} \right] = 0. \quad (147)$$

We will demonstrate that this sum vanishes for the first several orders in $M \ll 1$. From Arfken [34] we have the following two equations,

$$P_\ell(x) = \sum_{k=0}^{\ell/2} \frac{(-1)^k}{2^n} \frac{(2\ell-2k)!}{(\ell-k)!(\ell-2k)!} \frac{x^{\ell-2k}}{k!}, \quad (148)$$

and

$$Q_\ell(x) = 2^\ell \sum_{s=0}^{\infty} \frac{(\ell+s)!(\ell+2s)!}{(2\ell+2s+1)!s!} x^{-2s-\ell-1}. \quad (149)$$

Using these equations we can expand each $F_{+, \ell}$ mode in powers of M . Doing this, we find

$$\begin{aligned} \frac{(2\ell+1)}{M} P_\ell \left(\frac{b}{M} \right) Q_\ell \left(\frac{b}{M} \right) &= \frac{1}{b} + \frac{M^2}{2b^3} \left[1 + \frac{1}{(2\ell+3)(2\ell-1)} \right] + \\ &\frac{3}{8} \frac{M^4}{b^5} \left[1 + \frac{2}{(2\ell+3)(2\ell-1)} + \frac{11}{(2\ell+5)(2\ell+3)(2\ell-1)(2\ell-3)} \right] + O(M^6). \end{aligned} \quad (150)$$

If we focus on the ℓ independent pieces of Eq. (150), then we notice that they are exactly the expansion of $(b^2 - M^2)^{-1/2}$. The ℓ dependent pieces are each part of a vanishing sum (see Appendix B). Therefore, these terms vanish upon summation, leaving us with no force, as we expect from [15].

VI. CONCLUSIONS

For scalar and electromagnetic charges we have extended to accelerated trajectories and generic smooth spacetimes and coordinate systems the mode-sum regularization developed previously for geodesic orbits in a Kerr or Schwarzschild background. In this broader arena and for massive particles on geodesic trajectories in generic spacetimes, the singular behavior of the retarded self-force in a Lorenz or smoothly related gauge retains the form

$$f_{\alpha}^{sing,\ell\pm} = \pm A_{\alpha}(\ell + 1/2) + B_{\alpha},$$

and we have obtained expressions for the regularization parameters A_{α} and B_{α} .

Acknowledgments

We are indebted to Sam Gralla for helpful conversations and to an anonymous referee for corrections and comments. This work was supported in part by NSF Grants PHY 1001515, PHY 1307429, and PHY 0970074.

Appendix A: The Regularization Parameters for Higher Spins

Here we write the explicit regularization parameters for the self-force on a point electric charge and a point mass (computed in a Lorenz gauge). We directly parallel the approach taken for the scalar charge.

1. Electromagnetic Regularization Parameters

Until the final equation of this section, we set the charge e to 1.

We begin by writing Eq. (126), but we keep only the leading and sub-leading terms

$$\mathbf{A}_{\hat{\alpha}}^{sing} = \frac{u_{\hat{\alpha}}}{\sqrt{\hat{S}_0}} - \frac{\left[u_{\hat{\alpha}} \zeta_{\hat{\gamma}\hat{\delta}\hat{\epsilon}} + a_{\hat{\alpha}} u_{\hat{\gamma}} (\eta_{\hat{\epsilon}\hat{\delta}} + u_{\hat{\epsilon}} u_{\hat{\delta}}) \right] x^{\hat{\epsilon}} x^{\hat{\delta}} x^{\hat{\gamma}}}{\hat{S}_0^{3/2}}. \quad (\text{A1})$$

We now transform to our curvilinear coordinates, $v_{\alpha} = \partial_{\alpha} x^{\hat{\mu}} v_{\hat{\mu}}$. Expanding about the position of the particle (which is the origin of both our RNC and our locally Cartesian

angular coordinates), we have

$$\begin{aligned}\partial_\alpha x^{\hat{\mu}} &= (\partial_\alpha x^{\hat{\mu}})_0 + (\partial_\delta \partial_\alpha x^{\hat{\mu}})_0 x^\delta + O(x^2) \\ \partial_\alpha x^{\hat{\mu}} &= (\partial_\alpha x^{\hat{\mu}})_0 + (\partial_\epsilon x^{\hat{\mu}} \Gamma^\epsilon_{\alpha\delta})_0 x^\delta + O(x^2),\end{aligned}\tag{A2}$$

where the subscript ‘0’ denotes the value of a quantity at the position of the particle at time $t = 0$.

Applying this coordinate transformation, we find

$$\mathbf{A}_\alpha^{sing} = \frac{u_\alpha}{\sqrt{S_0}} + \frac{\zeta_{\alpha\gamma\delta\epsilon} x^\gamma x^\delta x^\epsilon}{S_0^{3/2}},\tag{A3}$$

where

$$\zeta_{\alpha\gamma\delta\epsilon} := (2u_\sigma \Gamma^\sigma_{\alpha\delta} - a_\alpha u_\delta) q_{\gamma\epsilon} - u_\alpha \zeta_{\delta\gamma\epsilon}.\tag{A4}$$

To calculate the regularization parameters for electromagnetism we use Eq. (112), written as

$$f_{EM}^{sing,\alpha} = e u^\beta g^{\alpha\sigma} [\nabla_\sigma \mathbf{A}_\beta^{sing} - \nabla_\beta \mathbf{A}_\sigma^{sing}] = u^\beta g^{\alpha\sigma} [\partial_\sigma \mathbf{A}_\beta^{sing} - \partial_\beta \mathbf{A}_\sigma^{sing}].$$

We now calculate the value of the individual modes of $\partial \mathbf{A}^{sing}$ in the limit that the field point approaches the source (i.e. as $\epsilon \rightarrow 0$). We then write the regularization parameters for the force as a linear combination of these.

From Eq. (A3), we have

$$\partial_\mu \mathbf{A}_{sing}^\alpha = -u^\alpha \frac{\partial_\mu S_0}{S_0^{3/2}} + \frac{\Lambda^\alpha_{\mu\beta\gamma\delta\epsilon} x^\beta x^\gamma x^\delta x^\epsilon}{S_0^{5/2}},\tag{A5}$$

where

$$\Lambda^\alpha_{\mu\beta\gamma\delta\epsilon} = 3\zeta^\alpha_{(\mu\beta\gamma)} q_{\delta\epsilon} - 3\zeta^\alpha_{\beta\gamma\delta} q_{\mu\epsilon}.\tag{A6}$$

In Eq. (A5), the leading order term is simply the four-velocity multiplied by the leading order term of the scalar field. We can therefore immediately evaluate the mode decomposition of this term,

$$\begin{aligned}A^\alpha_\mu L &= \left[u^\alpha \frac{-\partial_\mu S_0}{S_0^{3/2}} \right]_\ell = u^\alpha \lim_{\delta r \rightarrow 0^\pm} \frac{L}{2\pi} \int d\cos(\theta) P_\ell(\cos(\theta)) \int d\phi \left[\frac{-\partial_\mu S_0}{S_0^{3/2}} \right] = u^\alpha A_\mu^{(scalar)} L \\ &= \mp \frac{L u^\alpha}{\sqrt{g_{yy}}} \left[\frac{q_{\mu r} - q_{\mu x} q_{xr} / q_{xx} - g_{\mu y} g_{yr} / g_{yy}}{\sqrt{g_{yy} \tilde{\gamma}^2 + \lambda(g_{yy} + \Gamma^2)}} \right],\end{aligned}\tag{A7}$$

where we have used Eq. (102).

Now, we define

$$\Lambda_{\mu XY Y}^{\alpha} = \Lambda_{\mu x y y}^{\alpha} + \Lambda_{\mu x y x y}^{\alpha} + \Lambda_{\mu x y y x}^{\alpha} + x \leftrightarrow y, \quad (\text{A8})$$

which we use to write (recalling $w = \beta^2(1 + \beta^2)^{-1}$)

$$\begin{aligned} B_{\mu}^{\alpha} &= \left[\frac{\Lambda_{\mu \beta \gamma \delta \epsilon}^{\alpha} x^{\beta} x^{\gamma} x^{\delta} x^{\epsilon}}{S_0^{5/2}} \right]_{\ell} = \lim_{\delta r \rightarrow 0^{\pm}} \frac{L}{2\pi} \int d\cos(\theta) P_{\ell}(\cos(\theta)) \int d\phi \left[\frac{\Lambda_{\mu \beta \gamma \delta \epsilon}^{\alpha} x^{\beta} x^{\gamma} x^{\delta} x^{\epsilon}}{S_0^{5/2}} \right] \\ &= \frac{2}{3\pi(1 + \beta^2)^{3/2} \beta^4 q_{yy}^{5/2}} \left(B_{\mu}^{(E),\alpha} \hat{E}(w) + B_{\mu}^{(K),\alpha} \hat{K}(w) \right), \end{aligned} \quad (\text{A10})$$

where we define

$$B^{(E),\alpha}_{\mu} = (1 + \beta^2)(2 + \beta^2) \Lambda_{\mu XY Y}^{\alpha} - 2 \left[(1 + 2\beta^2) \Lambda_{\mu xxx x}^{\alpha} + (1 + \beta^2)^2 (1 - \beta^2) \Lambda_{\mu y y y y}^{\alpha} \right], \quad (\text{A11})$$

and

$$B^{(K),\alpha}_{\mu} = (2 + 3\beta^2) \Lambda_{\mu xxx x}^{\alpha} + (1 + \beta^2) \left[(2 - \beta^2) \Lambda_{\mu y y y y}^{\alpha} - 2 \Lambda_{\mu XY Y}^{\alpha} \right]. \quad (\text{A12})$$

We have cast Eqs. (A10), (A11) and (A12), into forms matching those of Eqs. (85), (86a), and (86b) for the scalar case. The sole differences are the presence of the additional raised index and the additional term in the definition of $\Lambda_{\mu \beta \gamma \delta \epsilon}^{\alpha}$. We will see similar symmetries between the scalar field and gravity in the next section.

Now we will write down the regularization parameters in terms of $A_{\alpha\mu}$ and $B_{\alpha\mu}$.

$$\begin{aligned} f_{\alpha}^{sing, EM\ell} &= [u^{\beta} (\partial_{\alpha} \mathbf{A}_{\beta} - \partial_{\beta} \mathbf{A}_{\alpha})]_{\ell} \\ f_{\alpha}^{sing, EM\ell} &= u^{\beta} [2A_{[\beta\alpha]} L + 2B_{[\beta\alpha]}]. \end{aligned} \quad (\text{A13})$$

Restoring the factors of the charge e , we find

$$A_{\alpha}^{(EM)} = 2e^2 u^{\beta} A_{[\beta\alpha]} \quad B_{\alpha}^{(EM)} = 2e^2 u^{\beta} B_{[\beta\alpha]}. \quad (\text{A14})$$

2. Gravitational Regularization Parameters

From Eq. (128), we can write the singular part of the trace-reversed metric perturbation as

$$\gamma_{\hat{\alpha}\hat{\beta}}^{sing} = \frac{4u_{\hat{\alpha}}u_{\hat{\beta}}}{\sqrt{\hat{S}_0}} - 4 \frac{\left[2u_{(\hat{\alpha}}a_{\hat{\beta})}u_{\hat{\epsilon}}q_{\hat{\delta}\hat{\gamma}} + u_{\hat{\alpha}}u_{\hat{\beta}}\zeta_{\hat{\epsilon}\hat{\delta}\hat{\gamma}} \right] x^{\hat{\epsilon}}x^{\hat{\delta}}x^{\hat{\gamma}}}{\hat{S}_0^{3/2}}. \quad (\text{A15})$$

We write this in terms of the actual metric perturbation, $h_{\mu\nu} = \gamma_{\mu\nu} - 1/2 g_{\mu\nu} \gamma_\mu^\mu$, and then apply the coordinate transformation to take us from RNCs to our curvilinear coordinates. Upon doing this, we find,

$$h_{sing}^{\alpha\beta} = 2 \frac{g^{\alpha\beta} + 2u^\alpha u^\beta}{\sqrt{S_0}} + \frac{\zeta_{\gamma\delta\epsilon}^{\alpha\beta} x^\gamma x^\delta x^\epsilon}{S_0^{3/2}}, \quad (\text{A16})$$

where

$$\zeta_{\gamma\delta\epsilon}^{\alpha\beta} := (8u^{(\alpha} a^{\beta)} u_\gamma - \partial_\gamma g^{\alpha\beta} + 4u^\sigma u^{(\alpha} \Gamma_{\sigma}^{\beta)}{}_\gamma) q_{\delta\epsilon} + (g^{\alpha\beta} + 2u^\alpha u^\beta) \zeta_{\gamma\delta\epsilon}. \quad (\text{A17})$$

We now compute $f_{GR}^{\alpha,sing}$ from Eq. (120),

$$\begin{aligned} f_{GR}^{\alpha,sing} &= -m q^{\alpha\delta} \left(\nabla_\beta h_{\gamma\delta}^{(s)} - \frac{1}{2} \nabla_\delta h_{\beta\gamma}^{(s)} \right) u^\beta u^\gamma \\ &= -m (g^{\alpha\delta} + u^\alpha u^\delta) u^\beta u^\gamma \left(\partial_\beta h_{\gamma\delta}^{sing} - \frac{1}{2} \partial_\delta h_{\beta\gamma}^{sing} - \Gamma_{\beta\gamma}^\mu h_{\mu\delta}^{sing} + \Gamma_{\delta[\gamma}^\mu h_{\beta]\mu}^{sing} \right). \end{aligned} \quad (\text{A18})$$

Therefore, we need to find the leading terms in the mode-sum decomposition of the metric perturbation and its derivative.

We first discuss the mode sum decomposition of the metric perturbation itself. Because the sub-leading term, is cubic in the coordinates $x^m u$ and is $O(\epsilon^0)$, its contribution will vanish. This means that the mode-sum decomposition of the metric perturbation evaluated at the position of the mass at time $t = 0$, is given by

$$\begin{aligned} h_{sing,\ell}^{\alpha\beta} &= 2 \lim_{\delta r \rightarrow 0^\pm} \frac{L}{2\pi} \int d\cos(\theta) P_\ell(\cos(\theta)) \int d\phi \left[\frac{g^{\alpha\beta} + 2u^\alpha u^\beta}{\sqrt{S_0}} \right] \\ h_{sing,\ell}^{\alpha\beta} &= B_{(h)}^{\alpha\beta} = 2 (g^{\alpha\beta} + 2u^\alpha u^\beta) \left[\frac{2}{\pi(1 + \beta^2)^{1/2}} \hat{K}(w) \right]. \end{aligned} \quad (\text{A19})$$

We use the subscript, (h) to distinguish $B_{(h)}^{\alpha\beta}$ from the quantity $B^{\alpha\beta}$ of the electromagnetism section above.

From Eq. (A16), we have

$$\partial_\mu h_{sing}^{\alpha\beta} = -(g^{\alpha\beta} + 2u^\alpha u^\beta) \frac{\partial_\mu S_0}{2S_0^{3/2}} + \frac{\Lambda_{\mu\gamma\delta\epsilon\sigma}^{\alpha\beta} x^\gamma x^\delta x^\epsilon x^\sigma}{S_0^{5/2}}, \quad (\text{A20})$$

where

$$\Lambda_{\mu\gamma\delta\epsilon\sigma}^{\alpha\beta} := \left[3\zeta_{(\mu\gamma\delta)}^{\alpha\beta} q_{\epsilon\sigma} - 3\zeta_{\gamma\delta\epsilon}^{\alpha\beta} q_{\mu\sigma} \right]. \quad (\text{A21})$$

The leading order term has the form

$$\begin{aligned}
A^{\alpha\beta}_{\mu} L &= \left[-(g^{\alpha\beta} + 2u^{\alpha}u^{\beta}) \frac{\partial_{\mu} S_0}{2S_0^{3/2}} \right]_{\ell} = -(g^{\alpha\beta} + 2u^{\alpha}u^{\beta}) \lim_{\delta r \rightarrow 0^{\pm}} \frac{L}{2\pi} \int d\cos(\theta) P_{\ell}(\cos(\theta)) \int d\phi \left[\frac{\partial_{\mu} S_0}{2S_0^{3/2}} \right] \\
&= (g^{\alpha\beta} + 2u^{\alpha}u^{\beta}) A_{\mu}^{(scalar)} L = A^{\alpha\beta}_{\mu} L = \mp \frac{L(g^{\alpha\beta} + 2u^{\alpha}u^{\beta})}{\sqrt{g_{yy}}} \left[\frac{g_{\mu r} + u_{\mu}u_r - \frac{(g_{\mu x} + u_{\mu}u_x)(g_{xr} + u_xu_r)}{g_{xx} + U_x^2} - \frac{g_{\mu y}g_{yr}}{g_{yy}}}{\sqrt{g_{yy}\tilde{\gamma}^2 + \lambda(g_{yy} + \Gamma^2)}} \right].
\end{aligned} \tag{A22}$$

Now, we define

$$\Lambda^{\alpha\beta}_{\mu XY Y} = \Lambda^{\alpha\beta}_{\mu xxyy} + \Lambda^{\alpha}_{\mu xyxy} + \Lambda^{\alpha}_{\mu xy yx} + x \leftrightarrow y, \tag{A23}$$

which allows us to write, (recalling $w = \beta^2(1 + \beta^2)^{-1}$)

$$\begin{aligned}
B^{\alpha\beta}_{\mu} &= \left[\frac{\Lambda^{\alpha\beta}_{\mu\sigma\gamma\delta\epsilon} x^{\sigma} x^{\gamma} x^{\delta} x^{\epsilon}}{S_0^{5/2}} \right]_{\ell} = \lim_{\delta r \rightarrow 0^{\pm}} \frac{L}{2\pi} \int d\cos(\theta) P_{\ell}(\cos(\theta)) \int d\phi \left[\frac{\Lambda^{\alpha\beta}_{\mu\sigma\gamma\delta\epsilon} x^{\sigma} x^{\gamma} x^{\delta} x^{\epsilon}}{S_0^{5/2}} \right] \\
&= \frac{2}{3\pi(1 + \beta^2)^{3/2}\beta^4 q_{yy}^{5/2}} \left(B^{(E),\alpha\beta}_{\mu} \hat{E}(w) + B^{(K),\alpha\beta}_{\mu} \hat{K}(w) \right),
\end{aligned} \tag{A24}$$

where we define

$$\begin{aligned}
B^{(E),\alpha\beta}_{\mu} &= -2 \left[(1 + 2\beta^2) \Lambda^{\alpha\beta}_{\mu xxxx} + (1 + \beta^2)^2 (1 - \beta^2) \Lambda^{\alpha\beta}_{\mu yyy y} \right] \\
&\quad + (1 + \beta^2) (2 + \beta^2) \Lambda^{\alpha\beta}_{\mu XXY Y},
\end{aligned} \tag{A25}$$

and

$$B^{(K),\alpha\beta}_{\mu} = (1 + \beta^2) \left[(2 - \beta^2) \Lambda^{\alpha\beta}_{\mu yyy y} - 2 \Lambda^{\alpha\beta}_{\mu XXY Y} \right] + (2 + 3\beta^2) \Lambda^{\alpha\beta}_{\mu xxxx}. \tag{A26}$$

We can now write the regularization parameters for gravity. From Eqs. (A18), (A19), (A22), and (A24), we see that only the partial derivatives of the metric perturbation contribute to $A_{(GR)}^{\alpha}$, allowing us to write,

$$A_{(GR)}^{\alpha} = -m (g^{\alpha\delta} + u^{\alpha}u^{\delta}) u^{\beta}u^{\gamma} \left(A_{\gamma\delta\beta} - \frac{1}{2} A_{\beta\gamma\delta} \right). \tag{A27}$$

The components $B_{(GR)}^{\alpha}$ are given by

$$B_{(GR)}^{\alpha} = -m (g^{\alpha\delta} + u^{\alpha}u^{\delta}) u^{\beta}u^{\gamma} \left(B_{\gamma\delta\beta} - \frac{1}{2} B_{\beta\gamma\delta} + \Gamma^{\mu}_{\delta[\gamma} B_{\beta]\mu}^{(h)} - \Gamma^{\mu}_{\beta\gamma} B_{\mu\delta}^{(h)} \right). \tag{A28}$$

We have obtained the explicit forms of the regularization parameters for all three spins in Eqs. (102) and (85) (scalar); (A14) (electromagnetism); and (A27) and (A28) (gravity). For all three spins, we have given the values in terms of ζ coefficients, which represent the numerator of the sub-leading terms of the potential (or perturbing metric), and Λ coefficients, which represent the numerator of the sub leading terms of the derivative of the potential (or perturbing metric).

Appendix B: Vanishing Sums

We show the relation

$$\sum_{\ell=0}^{\infty} \prod_{j=0}^N \frac{1}{(2\ell+1-2m_j)(2\ell+1+2m_j)} = 0, \quad (\text{B1})$$

for N and each m_j positive integers with the m_i distinct: $m_i \neq m_j, \forall i \neq j$.

The product in Eq. (B1) has a partial fraction decomposition of the form

$$\prod_{j=0}^N \frac{1}{(2\ell+1-2m_j)(2\ell+1+2m_j)} = \sum_{j=0}^N A_j \left[\frac{1}{(2\ell+1-2m_j)} - \frac{1}{(2\ell+1+2m_j)} \right], \quad (\text{B2})$$

where

$$A_i = \left[4m_i \prod_{j \neq i}^N [4(m_i^2 - m_j^2)] \right]^{-1}. \quad (\text{B3})$$

Eq. (B3) follows quickly from the decomposition $\frac{1}{(x-m)(x+m)} = \frac{1}{4m} \left[\frac{1}{x-2m} - \frac{1}{x+2m} \right]$. Because the sum in Eq. (B1) converges absolutely, we can re-order the sums over ℓ and j , writing

$$\sum_{\ell=0}^{\infty} \prod_{j=0}^N \frac{1}{(2\ell+1-2m_j)(2\ell+1+2m_j)} = \sum_{j=0}^N A_j \sum_{\ell=0}^{\infty} \left[\frac{1}{2\ell+1-2m_j} - \frac{1}{2\ell+1+2m_j} \right]. \quad (\text{B4})$$

We now show that the sum over ℓ vanishes for any positive integer m_j . We start by noting that the first $2m_j$ terms involving $1/(2\ell+1-2m_j)$ separately sum to zero (the terms are antisymmetric about $\ell = m_j - 1/2$):

$$\begin{aligned} \sum_{\ell=0}^{2m_j-1} \frac{1}{2\ell+1-2m_j} &= \left(\sum_{\ell=0}^{m_j-1} + \sum_{\ell=m_j}^{2m_j-1} \right) \frac{1}{2\ell+1-2m_j} \\ &= \sum_{\ell=0}^{m_j-1} \frac{1}{2\ell+1-2m_j} - \sum_{\ell'=0}^{m_j-1} \frac{1}{2\ell'+1-2m_j} = 0, \end{aligned} \quad (\text{B5})$$

where $\ell' = 2m_j - 1 - \ell$.

The remaining terms $1/(2\ell + 1 - 2m_j)$, beginning at $\ell = 2m_j$, are now identical to, and cancel, the terms $1/(2\ell + 1 + 2m_j)$, beginning at $\ell = 0$. Denoting by $\Theta(\ell - m_j)$ the step function vanishing for $\ell < m_j$, and having the value 1 for $\ell \geq m_j$, we have

$$\begin{aligned} \sum_{\ell=0}^{\infty} \left[\frac{1}{2\ell + 1 - 2m_j} - \frac{1}{2\ell + 1 + 2m_j} \right] &= \sum_{\ell=0}^{\infty} \left[\frac{\Theta(\ell - m_j)}{2\ell + 1 - 2m_j} - \frac{1}{2\ell + 1 + 2m_j} \right] \\ &= \sum_{\ell=0}^{\infty} \left[\frac{1}{2\ell + 1 + 2m_j} - \frac{1}{2\ell + 1 + 2m_j} \right] = 0. \quad \square \quad (\text{B6}) \end{aligned}$$

Appendix C: Regularization Parameters in the Original Background Coordinates

In Sects. II and IV, the components of the regularization parameters are obtained along a basis associated with locally Cartesian angular coordinates (LCAC); and the value we obtain for the vector B_α relies on extending the components of $q_{\alpha\beta}$ and u^α away from the particle by requiring that their components in the LCAC basis assume the values they take at the particle. For many applications, it is more useful to evaluate the components of A_α and B_α in the original coordinate system, as first done by Barack and Ori [2] and then later explained more completely in an appendix by Barack [3]. In this appendix, we follow the latter treatment and freeze the components of u^α and $q_{\alpha\beta}$ in the original t, r, θ, ϕ coordinates.

We define $(\tilde{x}^\alpha) = (\delta t = t, \delta r = r - r_0, \delta\theta = \theta - \theta_0, \delta\phi = \phi - \phi_0)$, so that \tilde{x}^μ agrees up to a constant with the original t, r, θ, ϕ coordinates; we continue to denote the locally Cartesian coordinates by $x^\alpha = (\delta t, \delta r, x, y)$. We denote by $\widetilde{W}_{\sigma\ldots\tau}^{\mu\ldots\nu}$ the components of a quantity $W_{\sigma\ldots\tau}^{\mu\ldots\nu}$, evaluated using the coordinate system x^μ . Note that the quantities $\zeta_{\mu\nu\lambda}$ and $\Lambda_{\mu\ldots\nu}$ involve partial derivatives of metric components and do not transform as tensors.

From the definitions of S_0 , S_1 , and the derivative of our singular field, (Eqs. (67), (68), and (42) respectively), we can write the components of the singular force in the original coordinates as

$$q^{-2} \tilde{f}_\mu^{sing} = -\frac{\tilde{q}_{\mu\nu} \tilde{x}^\nu}{\tilde{S}_0^{3/2}} + \frac{3\tilde{\zeta}_{\gamma\delta\epsilon} q_{\mu\nu} - (2\tilde{\zeta}_{\mu\gamma\delta} + \tilde{\zeta}_{\gamma\delta\mu}) \tilde{q}_{\nu\epsilon}}{\tilde{S}_0^{5/2}} \tilde{x}^\nu \tilde{x}^\epsilon \tilde{x}^\gamma \tilde{x}^\delta + O(\epsilon^0). \quad (\text{C1})$$

We still want to use the LCAC to simplify our integrations, retaining the \tilde{x}^μ components

$\widetilde{W}_{\sigma\dots\tau}^{\mu\dots\nu}$ of each quantity, but expressing them in terms of the LCAC. To do so, we write

$$\begin{aligned}\tilde{x}^3 &= \delta\theta = x^3 + \frac{1}{2}\cot(\theta_0)(x^4)^2 + O(\epsilon^3) \\ \tilde{x}^4 &= \delta\phi = \sin(\theta_0)^{-1}(x^4 - \cot(\theta_0)x^3x^4) + O(\epsilon^3)\end{aligned}\tag{C2}$$

(equivalent to Eq. (A.17) of [3]). Then

$$\tilde{x}^\alpha = a_\beta^\alpha x^\beta + c_{\beta\gamma}^\alpha x^\beta x^\gamma + O(\epsilon^3),\tag{C3}$$

where $a_\beta^\alpha = \partial_\beta \tilde{x}^\alpha|_0$, and $c_{\beta\gamma}^\alpha = \partial_\beta \partial_\gamma \tilde{x}^\alpha|_0$. By the arguments laid down before, it is clear that the higher order terms will give contributions to the self-force that either vanish at the particle or contribute to an order-unity term that vanishes upon integration over ϕ . Note that, at linear order, the transformation (C3) just replaces each occurrence of \tilde{x}^4 by $x^4/\sin\theta_0$.

The leading term acquires a first order correction:

$$\tilde{f}_\mu^{sing,L} = -\frac{\tilde{q}_{\mu\nu}a_\lambda^\nu x^\lambda}{(\tilde{q}_{\alpha\beta}a_\sigma^\alpha a_\tau^\beta x^\sigma x^\tau)^{3/2}} + \frac{(3\tilde{q}_{\mu\nu}\tilde{q}_{\iota\kappa} - \tilde{q}_{\mu\iota}\tilde{q}_{\nu\kappa})c_{\sigma\tau}^\iota x^\nu x^\kappa x^\sigma x^\tau}{(\tilde{q}_{\alpha\beta}a_\sigma^\alpha a_\tau^\beta x^\sigma x^\tau)^{5/2}}\tag{C4}$$

We take the mode-sum expansion of the force and evaluate these individual modes in the limit that $\epsilon \rightarrow 0$. The leading term will now give us the A_α term as before, and in the original coordinates we merely pick up an additional factor of $\sin\theta_0$;

$$\tilde{A}_{\alpha\pm} = \mp \sin\theta_0 q^2 \frac{\tilde{q}_{\alpha r} - \tilde{q}_{\alpha\theta}\tilde{q}_{\theta r}/\tilde{q}_{\theta\theta} - \tilde{q}_{\alpha\phi}\tilde{q}_{\phi r}/\tilde{q}_{\phi\phi}}{(\tilde{q}_{\theta\theta}\tilde{q}_{\phi\phi}\tilde{q}_{rr} - \tilde{q}_{\phi\phi}\tilde{q}_{\theta r}^2 - \tilde{q}_{\theta\theta}\tilde{q}_{\phi r}^2)^{1/2}}.\tag{C5}$$

For \tilde{B}_α , we evaluate the integral

$$\tilde{B}_\alpha = \frac{q^2}{2\pi} \tilde{P}_{\alpha\mu\nu\sigma\tau} \tilde{I}^{\mu\nu\sigma\tau},\tag{C6}$$

where

$$\tilde{I}^{\mu\nu\sigma\tau} = \lim_{\delta r \rightarrow 0} \int_0^{2\pi} d\phi \left[\frac{a_\alpha^\mu a_\beta^\nu a_\gamma^\sigma a_\delta^\tau x^\alpha x^\beta x^\gamma x^\delta}{(\tilde{q}_{\kappa\lambda} a_\epsilon^\kappa a_\iota^\lambda x^\epsilon x^\iota)^{5/2}} \right],\tag{C7}$$

and

$$\tilde{P}_{\alpha\mu\nu\gamma\delta} = 3\tilde{q}_{\alpha\delta}\tilde{\zeta}_{\mu\nu\gamma} - \tilde{q}_{\gamma\delta} \left(2\tilde{\zeta}_{\alpha\mu\nu} + \tilde{\zeta}_{\mu\nu\alpha} \right) + (3\tilde{q}_{\alpha\mu}\tilde{q}_{\epsilon\nu} - \tilde{q}_{\alpha\epsilon}\tilde{q}_{\mu\nu}) c_{\gamma\delta}^\epsilon,\tag{C8}$$

where $c_{\gamma\delta}^\epsilon$ is defined in Eq. (C3), whose only non-vanishing components are $c_{\phi\phi}^\theta = 4^{-1}\sin(2\theta_0)$ and $c_{\theta\phi}^\phi = c_{\phi\theta}^\phi = -2^{-1}\cot(\theta_0)$.

Notice that this equation is identical to Eq. (58) from [3], with the sole exception that we have included the acceleration in our $\tilde{\zeta}_{\alpha\beta\gamma}$. The limit in Eq. (C7) means that the integral

$I^{\mu\nu\gamma\delta}$ vanishes except when the indices only run over the (θ, ϕ) coordinates. Adopting the notation from [3], we let lowercase roman indices run over only θ and ϕ . Barack writes down the solutions to these integrals in Eqs. (48-57) [3], which we reproduce below. First, we define

$$\alpha = \sin^2(\theta_0)\tilde{q}_{\theta\theta}/\tilde{q}_{\phi\phi} - 1, \quad \tilde{\beta} = 2\sin(\theta_0)\tilde{q}_{\theta\phi}/\tilde{q}_{\phi\phi}. \quad (\text{C9})$$

Then, I^{abcd} is given by

$$I^{abcd} = \frac{\sin(\theta_0)^{5-N}}{(\alpha^2 + \tilde{\beta}^2)^2(4\alpha + 4 - \tilde{\beta}^2)^{3/2}(Q/2)^{1/2}} \left[Q I_K^{(N)} \hat{K}(\omega) + I_E^{(N)} \hat{E}(\omega) \right], \quad (\text{C10})$$

where

$$Q = \alpha + 2 - (\alpha^2 + \tilde{\beta}^2)^{1/2}, \quad \omega = \frac{2(\alpha^2 + \tilde{\beta}^2)^{1/2}}{\alpha + 2 + (\alpha^2 + \tilde{\beta}^2)^{1/2}}, \quad (\text{C11})$$

and $N = \delta_\phi^a + \delta_\phi^b + \delta_\phi^c + \delta_\phi^d$.

The ten quantities $I_K^{(N)}$ and $I_E^{(N)}$ are given by

$$\begin{aligned} I_K^{(0)} &= 4 \left[12\alpha^3 + \alpha^2(8 - 3\tilde{\beta}^2) - 4\alpha\tilde{\beta}^2 + \tilde{\beta}^2(\tilde{\beta}^2 - 8) \right], \\ I_E^{(0)} &= -16 \left[8\alpha^3 + \alpha^2(4 - 7\tilde{\beta}^2) + \alpha\tilde{\beta}^2(\tilde{\beta}^2 - 4) - \tilde{\beta}^2(\tilde{\beta}^2 + 4) \right], \end{aligned} \quad (\text{C12})$$

$$\begin{aligned} I_K^{(1)} &= 8\tilde{\beta} \left[9\alpha^2 - 2\alpha(\tilde{\beta}^2 - 4) + \tilde{\beta}^2 \right], \\ I_E^{(1)} &= -4\tilde{\beta} \left[12\alpha^3 - \alpha^2(\tilde{\beta}^2 - 52) + \alpha(32 - 12\tilde{\beta}^2) + \tilde{\beta}^2(3\tilde{\beta}^2 + 4) \right], \end{aligned} \quad (\text{C13})$$

$$\begin{aligned} I_K^{(2)} &= -4 \left[8\alpha^3 - \alpha^2(\tilde{\beta}^2 - 8) - 8\alpha\tilde{\beta}^2 + \tilde{\beta}^2(3\tilde{\beta}^2 - 8) \right], \\ I_E^{(2)} &= 8 \left[4\alpha^4 + \alpha^3(\tilde{\beta}^2 + 12) + \alpha(\tilde{\beta}^2 - 4)(3\tilde{\beta}^2 - 2\alpha) + 2\tilde{\beta}^2(3\tilde{\beta}^2 - 4) \right], \end{aligned} \quad (\text{C14})$$

$$\begin{aligned} I_K^{(3)} &= 8\tilde{\beta} \left[\alpha^3 - 7\alpha^2 + \alpha(3\tilde{\beta}^2 - 8) + \tilde{\beta}^2 \right], \\ I_E^{(3)} &= -4\tilde{\beta} \left[8\alpha^4 - 4\alpha^3 + \alpha^2(15\tilde{\beta}^2 - 44) + 4\alpha(5\tilde{\beta}^2 - 8) + \tilde{\beta}^2(3\tilde{\beta}^2 + 4) \right], \end{aligned} \quad (\text{C15})$$

$$\begin{aligned} I_K^{(4)} &= -4 \left[4\alpha^4 - 4\alpha^3 + \alpha^2(7\tilde{\beta}^2 - 8) + 12\alpha\tilde{\beta}^2 - \tilde{\beta}^2(\tilde{\beta}^2 - 8) \right], \\ I_E^{(4)} &= 16 \left[4\alpha^5 + 4\alpha^4 + \alpha^3(7\tilde{\beta}^2 - 4) + \alpha^2(11\tilde{\beta}^2 - 4) + (2\alpha + 1)\tilde{\beta}^2(\tilde{\beta}^2 + 4) \right]. \end{aligned} \quad (\text{C16})$$

1. The Regularization Parameters for Electromagnetism and Gravity

First, recall Eq. (129), reproduced below:

$$\begin{aligned} f_\mu^{s=1,sing} &= (\delta_\mu^\beta u^\alpha - \delta_\mu^\alpha u^\beta) \nabla_\beta A_\alpha^{sing} \\ f_\mu^{s=2,sing} &= (q_\mu^\beta (q^{\gamma\delta} + u^\gamma u^\delta) - 4q_\mu^\delta u^\beta u^\gamma) \nabla_\beta \frac{\gamma_{\gamma\delta}^{sing}}{4}. \end{aligned}$$

Since we have shown that only the leading and subleading terms in the singular vector potential and metric perturbation will give a non-vanishing contribution to the mode-sum when evaluated at the particle, this allows us to write the expressions for the singular vector potential and metric perturbation in a very convenient form, (taking the charge and mass to be unity)

$$\begin{aligned} \mathbf{A}_{\hat{\alpha}}^{sing} &= [u_{\hat{\alpha}} - a_{\hat{\alpha}} u_\nu x^\nu + O(\epsilon^1)] \Phi^{sing} \\ \frac{1}{4} \gamma_{\hat{\alpha}\hat{\beta}}^{sing} &= [u_{\hat{\alpha}} u_{\hat{\beta}} - 2a_{(\hat{\alpha}} u_{\hat{\beta})} u_\nu x^\nu + O(\epsilon^1)] \Phi^{sing}. \end{aligned} \quad (C17)$$

We transform from the RNC basis to the coordinate basis using Eq. (66), and plug in our expression for $\Phi^{sing} = S_0^{-1/2} - S_1(2S_0^{3/2})^{-1} + O(\epsilon^1)$, we find that the singular force for spin $s = 0, 1, 2$ can be written as

$$\tilde{f}_\alpha^{s,sing} = (-1)^s (q_s)^2 \left[-\frac{\tilde{q}_{\alpha\nu} \tilde{x}^\nu}{\tilde{S}_0^{3/2}} + \frac{\tilde{P}_{\alpha\mu\nu\gamma\delta}^s \tilde{x}^\mu \tilde{x}^\nu \tilde{x}^\gamma \tilde{x}^\delta}{\tilde{S}_0^{5/2}} + O(\epsilon^0) \right], \quad (C18)$$

where q_s is q, e, m for $s = 0, 1, 2$ respectively, and $P_{\alpha\mu\nu\gamma\delta}^s$ is given by

$$\tilde{P}_{\alpha\mu\nu\gamma\delta}^s = (\delta_{s,0} \delta_\alpha^\beta + \tilde{q}_\alpha^\beta (1 - \delta_{s,0})) \left(\tilde{P}_{\beta\mu\nu\gamma\delta} + s^2 \tilde{a}_\beta \tilde{q}_{\mu\nu} \tilde{q}_{\gamma\delta} + s \tilde{q}_{\beta\gamma} \tilde{u}^\lambda \tilde{u}^\rho \partial_\delta \tilde{g}_{\lambda\rho} \tilde{q}_{\mu\nu} \right), \quad (C19)$$

where $P_{\beta\mu\nu\gamma\delta}$ is defined in Eq. (C8). Thus, we can write the regularization parameters for spins 0,1, and 2:

$$\tilde{A}_{\alpha\pm}^s = \mp \sin(\theta_0) q_s^2 (-1)^s \frac{\tilde{q}_{\alpha r} - \tilde{q}_{\alpha\theta} \tilde{q}_{\theta r} / \tilde{q}_{\theta\theta} - \tilde{q}_{\alpha\phi} \tilde{q}_{\phi r} / \tilde{q}_{\phi\phi}}{(\tilde{q}_{\theta\theta} \tilde{q}_{\phi\phi} \tilde{q}_{rr} - \tilde{q}_{\phi\phi} \tilde{q}_{\theta r}^2 - \tilde{q}_{\theta\theta} \tilde{q}_{\phi r}^2)^{1/2}}, \quad (C20)$$

and

$$\tilde{B}_\alpha^s = (-1)^s \frac{q_s^2}{2\pi} \tilde{P}_{\alpha\mu\nu\gamma\delta}^s \tilde{I}^{\mu\nu\gamma\delta}, \quad (C21)$$

where $I^{\mu\nu\gamma\delta}$ is given in Eq. (C7).

Eqs. (C20) and (C21) simplify exactly to Eqs. (39-44) given in [3], when we take the geodesic limit, and specialize to a Kerr geometry.

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